# Mathematics I Chapter

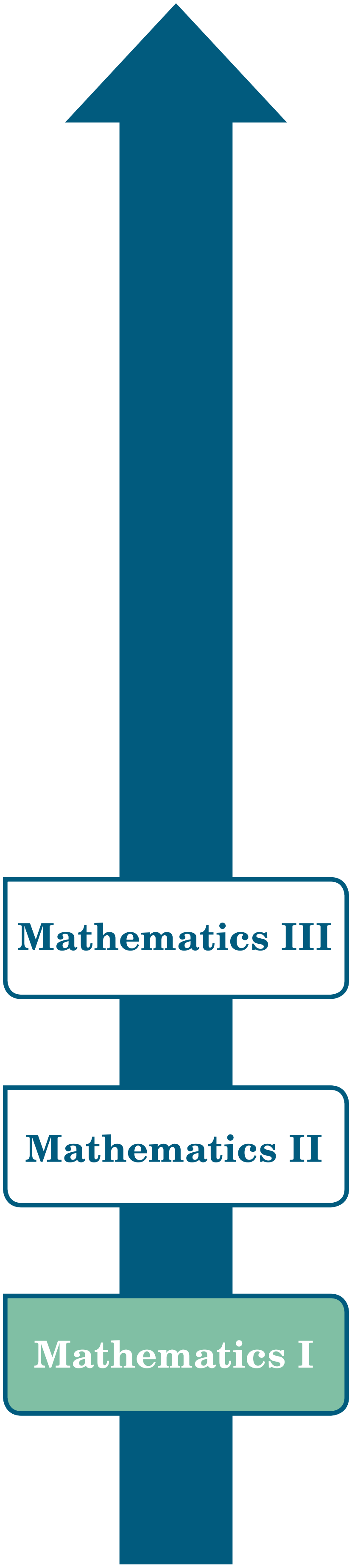
**of the**

# ***Mathematics Framework***

*for California Public Schools:*
*Kindergarten Through Grade Twelve*

**Adopted by the California State Board of Education, November 2013**

**Published by the California Department of Education**
**Sacramento, 2015**




# Mathematics I

The fundamental purpose of the Mathematics I course is to formalize and extend students' understanding of linear functions and their applications. The critical topics of study deepen and extend understanding of linear relationships—in part, by contrasting them with exponential phenomena and, in part, by applying linear models to data that exhibit a linear trend. Mathematics I uses properties and theorems involving congruent figures to deepen and extend geometric knowledge gained in prior grade levels. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The content of the course is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

## What Students Learn in Mathematics I

Students in Mathematics I continue their work with expressions and modeling and analysis of situations. In previous grade levels, students informally defined, evaluated, and compared functions, using them to model relationships between quantities. In Mathematics I, students learn function notation and develop the concepts of *domain* and *range*. Students move beyond viewing functions as processes that take inputs and yield outputs and begin to view functions as objects that can be combined with operations (e.g., finding $(f+g)(x)=f(x)+g(x)$). They explore many examples of functions, including sequences. They interpret functions that are represented graphically, numerically, symbolically, and verbally, translating between representations and understanding the limitations of various representations. They work with functions given by graphs and tables, keeping in mind that these representations are likely to be approximate and incomplete, depending upon the context. Students' work includes functions that can be described or approximated by formulas, as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They also interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

Students who are prepared for Mathematics I have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Mathematics I builds on these earlier experiences by asking students to analyze and explain the process of solving an equation and to justify the process used in solving a system of equations. Students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and using them to solve problems. They master solving linear equations and apply related solution techniques and the laws of exponents to the creation and solving of simple exponential equations. Students explore systems of equations and inequalities, finding and interpreting solutions. All of this work is based on understanding quantities and the relationships between them.

In Mathematics I, students build on their prior experiences with data, developing more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

In previous grade levels, students were asked to draw triangles based on given measurements. They also gained experience with rigid motions (translations, reflections, and rotations) and developed notions about what it means for two objects to be congruent. In Mathematics I, students establish triangle congruence criteria based on analyses of rigid motions and formal constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why the constructions work. Finally, building on their work with the Pythagorean Theorem in the grade-eight standards to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

## Examples of Key Advances from Kindergarten Through Grade Eight

- Students build on previous work with solving linear equations and systems of linear equations from grades seven and eight in two ways: (a) They extend to more formal solution methods, including attending to the structure of linear expressions; and (b) they solve linear inequalities.
- Students' work with patterns and number sequences in the early grades extends to an understanding of sequences as functions.
- Students formalize their understanding of the definition of a function, particularly their understanding of linear functions, emphasizing the structure of linear expressions. Students also begin to work with exponential functions by comparing them to linear functions.
- Work with congruence and similarity transformations that started in grades six through eight progresses. Students consider sufficient conditions for the congruence of triangles.
- Work with bivariate data and scatter plots in grades six through eight is extended to working with lines of best fit (Partnership for Assessment of Readiness for College and Careers [PARCC] 2012, 26).

## Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the MP standards. Connecting practices and content happens in the context of *working on problems;* the very first MP standard is to make sense of problems and persevere in solving them. Table M1-1 gives examples of how students can engage in the MP standards in Mathematics I.

**Table M1-1. Standards for Mathematical Practice—Explanation and Examples for Mathematics I**

| Standards for Mathematical Practice | Explanation and Examples |
|---|---|
| **MP.1**<br>Make sense of problems and persevere in solving them. | Students persevere when attempting to understand the differences between linear and exponential functions. They make diagrams of geometric problems to help make sense of the problems. |
| **MP.2**<br>Reason abstractly and quantitatively. | Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. |
| **MP.3**<br>Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only). | Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If ________, then ________" when explaining their solution methods and provide justification for their reasoning. |
| **MP.4**<br>Model with mathematics. | Students apply their mathematical understanding of linear and exponential functions to many real-world problems, such as linear and exponential growth. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts. |
| **MP.5**<br>Use appropriate tools strategically. | Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the results. |
| **MP.6**<br>Attend to precision. | Students use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. |
| **MP.7**<br>Look for and make use of structure. | Students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. |
| **MP.8**<br>Look for and express regularity in repeated reasoning. | Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number $m$. Therefore, if $(x, y)$ is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line. |

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. In the description of the Mathematics I content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum.

Examples of places where specific Mathematical Practice standards can be implemented in the Mathematics I standards are noted in parentheses, with the standard(s) also listed.

## Mathematics I Content Standards, by Conceptual Category

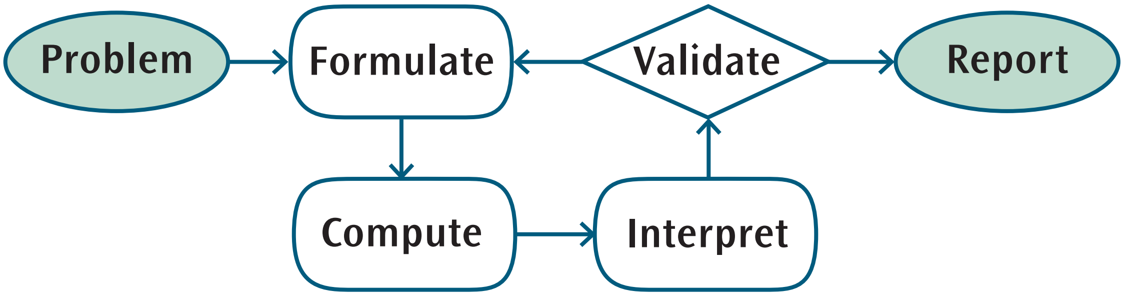
The Mathematics I course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics I are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

### Conceptual Category: Modeling

Throughout the CA CCSSM, specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known and unknown? Can a table of data be made? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. In addition, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure M1-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

**Figure M1-1. The Modeling Cycle**




The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding linear and exponential functions, graphing, solving equations, and rates of change are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

## Conceptual Category: Functions

The standards in the Functions conceptual category can serve as motivation for the study of standards in the other Mathematics I conceptual categories. For instance, an equation wherein one is asked to "solve for $x$" can be seen as a search for the input of a function $f$ that gives a specified output, and solving the equation amounts to undoing the work of the function. Or, the graph of an equation such as $y=\frac{1}{3}x+5$ can be seen as a representation of a function $f$ where $f(x)=\frac{1}{3}x+5$. Solving a more complicated equation can be seen as asking, "For which values of $x$ do two functions $f$ and $g$ agree? (i.e., when does $f(x)=g(x)$?)," and the intersection of the two graphs $y=f(x)$ and $y=g(x)$ is then connected to the solution of this equation. In general, functions describe in a precise way how two quantities are related and can be used to make predictions and generalizations, keeping true to the emphasis on modeling in higher mathematics.

Functions describe situations in which one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because theories are continually formed about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models. In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, $v$; the rule $T(v)=\frac{100}{v}$ expresses this relationship algebraically and defines a function whose name is $T$.

The set of inputs to a function is called its *domain*. The domain is often assumed to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. When relationships between quantities are described, the defining characteristic of a function is that the input value determines the output value, or equivalently, that the output value depends upon the input value (University of Arizona [UA] Progressions Documents for the Common Core Math Standards 2013c, 2).

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; by an assignment, such as the fact that each individual is given a unique Social Security Number; by an algebraic expression, such as $f(x)=a+bx$ ; or by a recursive rule, such as $f(n+1)=f(n)+b$, $f(0)=a$. The graph of a function is often a useful way of visualizing the relationship that the function models, and manipulating a mathematical expression for a function can shed light on the function's properties.

## Interpreting Functions F-IF

**Understand the concept of a function and use function notation.** [Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If $f$ is a function and $x$ is an element of its domain, then $f(x)$ denotes the output of $f$ corresponding to the input $x$. The graph of $f$ is the graph of the equation $y=f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0)=f(1)=1$, $f(n+1)=f(n)+f(n-1)$ *for* $n \geq 1$.

**Interpret functions that arise in applications in terms of the context.** [Linear and exponential (linear domain)]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

While the grade-eight standards called for students to work informally with functions, students in Mathematics I begin to refine their understanding and use the formal mathematical language of functions. Standards **F-IF.1–9** deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (**MP.6**). Standard **F-IF.3** calls for students to learn the language of functions and that a function has a domain that must be specified as well as a corresponding range. For instance, by itself, the equation $f(x)=2^x$ does not describe a function entirely. Similarly, though the expressions in the equations $f(x)=3x-4$ and $g(n)=3n-4$

look the same, except for the variables used, $f$ may have as its domain all real numbers, while $g$ may have as its domain the natural numbers (i.e., $g$ defines a sequence). Students make the connection between the graph of the equation $y = f(x)$ and the function itself—namely, that the coordinates of any point on the graph represent an input and output, expressed as $(x, f(x))$—and understand that the graph is a *representation* of a function. They connect the domain and range of a function to its graph (**F-IF.5**). Note that there is neither an exploration of the notion of *relation versus function* nor the *vertical line test* in the CA CCSSM. This is by design. The core question when students investigate functions is, "Does each element of the domain correspond to exactly one element of the range?" (UA Progressions Documents 2013c, 8).

Standard **F-IF.3** introduces sequences as functions. In general, a sequence is a function whose inputs consist of a subset of the integers, such as $\{0, 1, 2, 3, 4, 5, \ldots\}$. Students can begin to study sequences in simple contexts, such as when calculating their total pay, $P$, when working for $n$ days at \$65 per day, obtaining a general expression $P(n) = 65 \bullet n$. Students investigate geometric sequences of the form $g(n) = ar^n$, $n \geq 1$, or $g(1) = ar$, $g(n+1) = r \bullet g(n)$, for $n \geq 2$, when they study population growth or decay, as in the availability of a medical drug over time, or financial mathematics, such as when determining compound interest. Notice that the domain is included in the description of the rule (adapted from UA Progressions Documents 2013c, 8).

## Interpreting Functions F-IF

**Analyze functions using different representations.** [Linear and exponential]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Standards **F-IF.7** and **F-IF.9** call for students to represent functions with graphs and identify key features in the graph. In Mathematics I, students study only linear, exponential, and absolute value functions. They represent the same function algebraically in different forms and interpret these differences in terms of the graph or context.

## Building Functions F-BF

**Build a function that models a relationship between two quantities.** [For F.BF.1, 2, linear and exponential (integer inputs)]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

**Build new functions from existing functions.** [Linear and exponential; focus on vertical translations for exponential.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Knowledge of functions and expressions is only part of the complete picture. One must be able to understand a given situation and apply function reasoning to model how quantities change together. Often, the function created sheds light on the situation at hand; one can make predictions of future changes, for example. This is the content of standards **F-BF.1** and **F-BF.2** (starred to indicate they are modeling standards). Mathematics I features the introduction of arithmetic and geometric sequences, written both explicitly and recursively. Students can often see the recursive pattern of a sequence—that is, how the sequence changes from term to term—but they may have a difficult time finding an explicit formula for the sequence.

For example, a population of cyanobacteria can double every 6 hours under ideal conditions, at least until the nutrients in its supporting culture are depleted. This means a population of 500 such bacteria would grow to 1000 in the first 6-hour period, to 2000 in the second 6-hour period, to 4000 in the third 6-hour period, and so on. So if $n$ represents the number of 6-hour periods from the start, the population at that time $P(n)$ satisfies $P(n)=2\bullet P(n-1)$. This is a *recursive* formula for the sequence $P(n)$, which gives the population at a given time period $n$ in terms of the population at time period $n-1$. To find a closed, *explicit* formula for $P(n)$, students can reason that

$$P(0)=500,\ P(1)=2\bullet 500,\ P(2)=2\bullet 2\bullet 500,\ P(3)=2\bullet 2\bullet 2\bullet 500,\ \ldots$$

A pattern emerges: that $P(n)=2^n\bullet 500$. In general, if an initial population $P_0$ grows by a factor $r>1$ over a fixed time period, then the population after $n$ time periods is given by $P(n)=P_0r^n$.

The following example shows that students can create functions based on prototypical ones.

| Example: Exponential Growth | F-BF.3 |
|---|---|

The following example illustrates the type of problem that students can solve after they have worked with basic exponential functions.

On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If the algae continue to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. Based on the current rate at which the algae are growing, this will happen on June 30.

### Possible Questions to Ask:

a. When will the lake be covered halfway?

b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae, as a function of time (in days) that passes since the algae were introduced into the lake.

### Solution and Comments:

a. Since the population doubles each day, and since the entire lake will be covered by June 30, this implies that half the lake was covered on June 29.

b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, one can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} \approx 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time $t$ is therefore $P(t) = \left(1.86 \times 10^{-7}\right)2^t$.

Adapted from Illustrative Mathematics 2013i.

It should be noted that sequences often *do not* lend themselves to compact, explicit formulas such as those in the preceding example. When provided with a sufficient number of examples, students will be able to see this. The means for deciding which sequences do have explicit formulas, such as arithmetic and geometric sequences, is an important area of instruction.

The content of standard **F-BF.3** has typically been left to later courses. In Mathematics I, the focus is on linear and exponential functions. Even and odd functions are addressed in later courses. In keeping with the theme of the input–output interpretation of a function, students should work toward developing an understanding of the effect on the output of a function under certain transformations, such as in the table below:

| Expression | Interpretation |
|---|---|
| $f(a+2)$ | The output when the input is 2 greater than $a$ |
| $f(a)+3$ | 3 more than the output when the input is $a$ |
| $2f(x)+5$ | 5 more than twice the output of $f$ when the input is $x$ |

Such understandings can help students to see the effect of transformations on the graph of a function, and in particular, can aid in understanding why it appears that the effect on the graph is the opposite to the transformation on the variable. For example, the graph $y = f(x+2)$ is the graph of $f$ shifted 2 to the left, not to the right (UA Progressions Documents 2013c, 7).

| **Linear, Quadratic, and Exponential Models** | **F-LE** |
|---|---|

**Construct and compare linear, quadratic, and exponential models and solve problems.** [Linear and exponential]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

**Interpret expressions for functions in terms of the situation they model.** [Linear and exponential of form $f(x) = b^x + k$]

5. Interpret the parameters in a linear or exponential function in terms of a context. ★

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. In standards **F-LE.1a–c,** students recognize and understand the defining characteristics of linear and exponential functions. Students have already worked extensively with linear equations. They have developed an understanding that an equation in two variables of the form $y = mx + b$ exhibits a special relationship between the variables $x$ and $y$—namely, that a change of $\Delta x$ in the variable $x$, the independent variable, results in a change of $\Delta y = m \bullet \Delta x$ in the dependent variable $y$. They have seen this informally, in graphs and tables of linear relationships, starting in the grade-eight standards (**8.EE.5, 8.EE.6, 8.F.3**). If one considers only integer values of $x$, so that the incremental change in $x$ is simply 1 unit, then the change in $y$ is exactly $m$; it is this constant rate of change, $m$, that defines linear relationships, both in discrete linear sequences and in general linear functions of one real variable. Stated in a different way, students recognize that for successive whole-number input values, $x$ and $x+1$, a linear function $f(x) = mx + b$ exhibits a constant rate of change:

$$f(x+1) - f(x) = [m(x+1)+b] - (mx+b) = m(x+1-x) = m$$

In contrast, exponential equations such as $g(n) = ab^n$ exhibit a constant percent change. For instance, a t-table for the equation $y = 3^n$ illustrates the constant ratio of successive $y$-values for this equation:

| | $= 3^n$ | Ratio of successive $y$-values |
|---|---|---|
| 1 | 3 | |
| 2 | 9 | $\frac{9}{3} = 3$ |
| 3 | 27 | $\frac{27}{9} = 3$ |
| 4 | 81 | $\frac{81}{27} = 3$ |

This table shows that each value of $y$ is 3 times the value preceding it (i.e., 300% of the value preceding it), illustrating the constant percent change of this exponential. In the general case, we have

$$\frac{g(n+1)}{g(n)} = \frac{ab^{n+1}}{ab^n} = \frac{b^{n+1}}{b^n} = b^{(n+1)-n} = b,$$

which illustrates the constant ratio of successive values of $g$.[1]

The standards require students to prove the result above for linear functions (**F-LE.1a**). In general, students must also be able to recognize situations that represent both linear and exponential functions and construct functions to describe the situations (**F-LE.2**). Finally, students interpret the parameters in linear and exponential functions and model physical problems with such functions.

A graphing utility, spreadsheet, or computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions (**MP.4** and **MP.5**). Real-world examples where this can be explored involve investments, mortgages, and other financial instruments. For example, students can develop formulas for annual compound interest based on a general formula, such as $P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$, where $P_0$ is the initial amount invested, $r$ is the interest rate, $n$ is the number of times the interest is compounded per year, and $t$ is the number of years the money is invested. They can explore values after different time periods and compare different rates and plans using computer algebra software or simple spreadsheets (**MP.5**). This hands-on experimentation with such functions helps students develop an understanding of the functions' behavior.

## Conceptual Category: Number and Quantity

In real-world problems, the answers are usually not numbers, but *quantities*: numbers with units, which involve measurement. In their work in measurement up through grade eight, students primarily measure commonly used attributes such as length, area, and volume. In higher mathematics, students encounter a wider variety of units in modeling—for example, when considering acceleration, currency

1. In Mathematics I of the CA CCSSM, only integer values for $x$ are considered in exponential equations such as $y = b^x$.

conversions, derived quantities such as person-hours and heating degree-days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages.

| **Quantities** | **N-Q** |
|---|---|

**Reason quantitatively and use units to solve problems.** [Foundation for work with expressions, equations, and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

In Mathematics I, students reason through problems with careful selection of units, and they use units to understand problems and make sense of the answers they deduce. Standards **N-Q.1–3** are modeling standards that refer to students' appropriate use of units and definition of quantities. For instance, students can evaluate the accuracy of the following conclusion made in a magazine:

> On average the human body is more than 50 percent water [by weight]. Runners and other endurance athletes average around 60 percent. This equals about 120 soda cans' worth of water in a 160-pound runner! (Illustrative Mathematics 2013p)

Students look for appropriate unit conversions. For example, a typical soda can holds 12 ounces of fluid, a pound is equivalent to 16 dry ounces, and an ounce of water weighs approximately 1 dry ounce (at the temperature of the human body).

## Conceptual Category: Algebra

In the Algebra conceptual category, students extend the work with expressions that they started in grades six through eight. They create and solve equations in context, utilizing the power of variable expressions to model real-world problems and solve them with attention to units and the meaning of the answers they obtain. They continue to graph equations, understanding the resulting picture as a representation of the points satisfying the equation. This conceptual category accounts for a large portion of the Mathematics I course and, along with the Functions category, represents the main body of content.

The Algebra conceptual category in higher mathematics is very closely related to the Functions conceptual category (UA Progressions Documents 2013b, 2):

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.

- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the expression is of the form $y = (\text{expression in } x)$ or if it can be put into that form by solving for $y$.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent, they define the same function.
- The solutions to an equation in one variable can be understood as the input values that yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing functions defined by each side and finding the points where the graphs intersect.

Thus, in light of understanding functions, the main content of the Algebra category (solving equations, working with expressions, and so forth) has a very important purpose.

## Seeing Structure in Expressions — A-SSE

**Interpret the structure of expressions.** [Linear expressions and exponential expressions with integer exponents]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of $P$ and a factor not depending on $P$.* ★

An expression can be viewed as a recipe for a calculation, with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price, $p$. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor. Students began this work in grades six and seven and continue this work with more complex expressions in Mathematics I.

The following example might arise in a modeling context. It emphasizes the importance of understanding the meaning of expressions in a given problem.

**Example** A-SSE.1

A company uses two different-sized trucks to deliver sand. The first truck can transport $x$ cubic yards, and the second truck can transport $y$ cubic yards. The first truck makes $S$ trips to a job site, while the second makes $T$ trips. What do the following expressions represent in practical terms?

a. $S+T$

b. $x+y$

c. $xS+yT$

d. $\frac{xS+yT}{S+T}$

**Solutions:**

a. $S+T$ = the total number of trips both trucks make to a job site.

b. $x+y$ = the total amount of sand that both trucks can transport together.

c. $xS+yT$ = the total amount of sand (in cubic yards) being delivered to a job site by both trucks.

d. $\frac{xS+yT}{S+T}$ = the average amount of sand being transported per trip.

## Creating Equations A-CED

**Create equations that describe numbers or relationships.** [Linear and exponential (integer inputs only); for A.CED.3, linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance $R$.* ★

An *equation* is a statement of equality between two expressions. The values that make the equation true are the solutions to the equation. An *identity,* in contrast, is true for all values of the variables; rewriting an expression in an equivalent form often creates an identity. The solutions of an equation in one variable form a set of numbers that can be plotted on a number line; the solutions of an equation in two variables form a set of ordered pairs of numbers that can be plotted in the coordinate plane. This set of standards (**A-CED.1–4**) calls for students to create equations to solve problems, correctly graph the equations on the coordinate plane, and interpret solutions in a modeling context.

The following example is designed to have students think about the meaning of the quantities presented in the context and choose which quantities are appropriate for the two different constraints presented. In particular, note that the purpose of the task is to have students *generate* the constraint equations for each part (although the problem statements avoid using this particular terminology), not *solve* the equations (Illustrative Mathematics 2013g).

**Example** **A-CED.2–3**

The arabica coffee variety yields about 750 kilograms of coffee beans per hectare, and the robusta coffee variety yields approximately 1200 kilograms per hectare. Suppose that a plantation has $a$ hectares of arabica and $r$ hectares of robusta.

a. Write an equation relating $a$ and $r$ if the plantation yields 1,000,000 kilograms of coffee.

b. On August 14, 2003, the world market price of coffee was \$1.42 per kilogram of arabica and \$0.73 per kilogram of robusta. Write an equation relating $a$ and $r$ if the plantation produces coffee worth \$1,000,000.

**Solution:**

a. The quantity $a$ hectares of arabica will yield $750a$ kilograms (kg) of beans, and $r$ hectares of robusta will yield $1200r$ kg of beans. So the constraint equation is

$$750a + 1200r = 1{,}000{,}000.$$

b. Since $a$ hectares of arabica yield $750a$ kg of beans worth \$1.42/kg, the total dollar value of $1.42(750a) = 1065a$. Likewise, $r$ hectares of robusta yield $1200r$ kg of beans worth \$0.73/kg, for a total dollar value of $0.73(1200r) = 876r$. So the equation governing the possible values of $a$ and $r$ coming from the total market value of the coffee is

$$1065a + 876r = 1{,}000{,}000.$$

One California addition to the Common Core State Standards for Mathematics is the creation of equations involving absolute values (**A-CED.1 [CA]**). The basic absolute value function has at least two useful definitions: (1) a descriptive, verbal definition and (2) a formula definition. A common definition of the absolute value of $x$ is

$$|x| = \text{the distance from the number } x \text{ to 0 (on a number line).}$$

An understanding of the number line easily yields that, for example, $|0| = 0$, $|7| = 7$, and $|-3.9| = 3.9$. However, an equally valid "formula" definition of absolute value reads as follows:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

In other words, $|x|$ is simply $x$ whenever $x$ is 0 or positive, but $|x|$ is the opposite of $x$ whenever $x$ is negative. Either definition can be extended to an understanding of the expression $|x-a|$ as the distance between $x$ and $a$ on a number line, an interpretation that has many uses. For a simple application of this idea, suppose a type of bolt is to be mass-produced in a factory with the specification that its width be 5 mm with an error no larger than 0.01 mm. If $w$ represents the width of a given bolt produced on the production line, then $w$ must satisfy the inequality $|w-5| \leq 0.01$; that is, the difference between the actual width $w$ and the target width should be less than or equal to 0.01 (**MP.4, MP.6**). Students should become comfortable with the basic properties of absolute values (e.g., $|x| + a \neq |x+a|$) and with solving absolute value equations and interpreting the solution.

In higher mathematics courses, intervals on the number line are often denoted by an inequality of the form $|x-a| \leq d$ for a positive number $d$. For example, $|x-2| \leq \frac{1}{2}$ represents the closed interval $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This can be seen by interpreting $|x-2| \leq \frac{1}{2}$ as "the distance from $x$ to 2 is less than or equal to $\frac{1}{2}$" and deciding which numbers fit this description.

On the other hand, in the case where $x-2<0$, the following would hold true: $|x-2| = -(x-2) \leq \frac{1}{2}$, so that $x \geq 1\frac{1}{2}$. In the case where $x-2 \geq 0$, $|x-2| = x-2 \leq \frac{1}{2}$, which means that $x \leq 2\frac{1}{2}$. Since students are looking for all values of $x$ that satisfy both inequalities, the interval is $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This shows how the formula definition can be used to find this interval.

## Reasoning with Equations and Inequalities A-REI

**Understand solving equations as a process of reasoning and explain the reasoning.** [Master linear; learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

**Solve equations and inequalities in one variable.**

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [Linear inequalities; literal equations that are linear in the variables being solved for; exponential of a form, such as $2^x = \frac{1}{16}$.]

**3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA**

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions. In Mathematics I, students solve linear equations and inequalities in one variable, including equations and inequalities with absolute values and equations with coefficients represented by letters (**A-REI.3, A REI.3.1**). When solving equations, students make use of the symmetric and transitive properties and particular properties of equality regarding operations (e.g., "Equals added to equals are equal"). Standard **A-REI.1** requires that in any situation, students can solve an equation *and explain the steps* as resulting from previous true equations and using the aforementioned properties (**MP.3**). In this way, the idea of *proof,* while not explicitly named, is given a prominent role in the solving of equations, and the reasoning and justification process is not simply relegated to a future mathematics course. The following example illustrates the justification process that may be expected in Mathematics I.

**On Solving Equations:** A written sequence of steps is code for a narrative line of reasoning that would use words such as *if, then, for all,* and *there exists.* In the process of learning to solve equations, students should learn certain "if–then" moves—for example, "If $x = y$, then $x + c = y + c$ for any $c$." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in this domain (**REI**) is that students understand that solving equations is a process of reasoning (**A-REI.1**).

### Fragments of Reasoning

$$
\begin{gathered}
2x - 5 = 16 - x \\
2x - 5 + x = 16 - x + x \\
3x - 5 = 16 \\
3x = 21 \\
x = \frac{21}{3} = 7
\end{gathered}
$$

This sequence of equations is shorthand for a line of reasoning: *"If twice a number minus 5 equals 16 minus that number, then three of that number minus 5 must be 16, by the properties of equality. But that means three times that number is 21, so the number is 7."*

Adapted from UA Progressions Documents 2013b, 13.

The same solution techniques used to solve equations can be used to rearrange formulas to highlight specific quantities and explore relationships between the variables involved. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1 + b_2}{2}\right)h$, can be solved for $h$ using the same deductive process (**MP.7, MP.8**). As will be discussed later, functional relationships can often be explored more deeply by rearranging equations that define such relationships; thus, the ability to work with equations that have letters as coefficients is an important skill.

**Solve systems of equations.** [Linear systems]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. The process of adding one equation to another is understood in this way: If the two sides of one equation are equal, and the two sides of another equation are equal, then the sum (or difference) of the left sides of the two equations is equal to the sum (or difference) of the right sides. The reversibility of these steps justifies that an equivalent system of equations has been achieved. This crucial point should be consistently noted when students reason about solving systems of equations (UA Progressions Documents 2013b, 11).

When solving systems of equations, students also make frequent use of the substitution property of equality—for example, when solving the system $2x-9y=5$ and $y=\frac{1}{3}x+1$, the expression $\frac{1}{3}x+1$ can be substituted for $y$ in the first equation to obtain $2x-9\left(\frac{1}{3}x+1\right)=5$. Students also solve such systems approximately, by using graphs and tables of values (**A-REI.5–6**). Presented in context, the method of solving a system of equations by elimination takes on meaning, as the following example shows.

**Example: Solving Simple Systems of Equations** **A-REI.6**

To get started with understanding how to solve systems of equations by linear combinations, students can be encouraged to interpret a system in terms of real-world quantities, at least in some cases. For instance, suppose one wanted to solve this system:

$$3x+y=40$$
$$4x+2y=58$$

Now consider the following scenario: Suppose 3 CDs and a magazine cost \$40, while 4 CDs and 2 magazines cost \$58.

- What happens to the price when you add 1 CD and 1 magazine to your purchase?
- What is the price if you decided to buy only 2 CDs and no magazine?

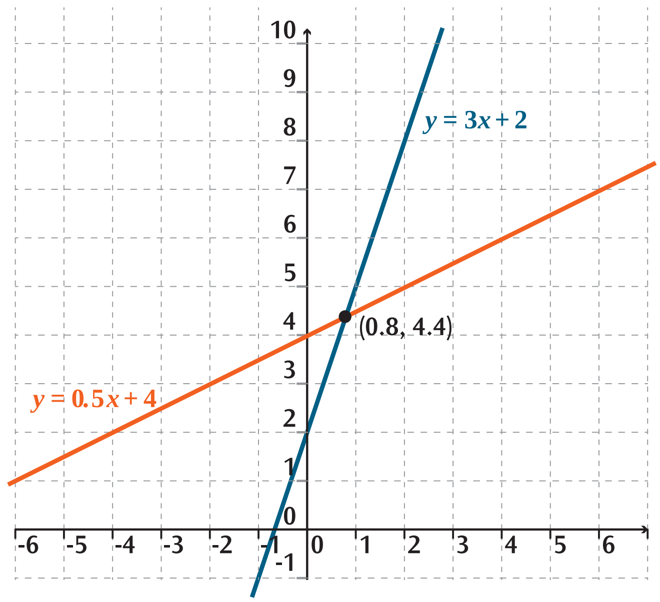
Answering these questions amounts to realizing that since $(3x+y)+(x+y)=40+18$, we must have that $x+y=18$. Therefore, $(3x+y)+(-1)(x+y)=40+(-1)18$, which implies that $2x=22$, or 1 CD costs \$11. The value of $y$ can now be found using either of the original equations: $y=7$.

**Represent and solve equations and inequalities graphically.** [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the $x$-coordinates of the points where the graphs of the equations $y=f(x)$ and $y=g(x)$ intersect are the solutions of the equation $f(x)=g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

One of the most important goals of instruction in mathematics is to illuminate connections between different mathematical concepts. In particular, standards **A-REI.10–12** call for students to learn the relationship between the algebraic representation of an equation and its graph plotted in the coordinate plane and understand geometric interpretations of solutions to equations and inequalities. As students become more comfortable with function notation after studying standards **F-IF.1–2** — for example, writing $f(x)=3x+2$ and $g(x)=-\frac{1}{2}x+4$ — they begin to see solving the equation $3x+2=-\frac{1}{2}x+4$ as solving the equation $f(x)=g(x)$. That is, they find those $x$-values where two functions take on the same output value. Moreover, they graph the two equations (see figure M1-2) and see that the $x$-coordinate(s) of the point(s) of intersection of the graphs of $y=f(x)$ and $y=g(x)$ are the solutions to the original equation.

**Figure M1-2. Graph of a System of Two Linear Equations**




Students also create tables of values for functions to approximate or find exact solutions to equations such as that above. For example, they may use spreadsheet software to construct a table (see table M1-2).

**Table M1-2. Values for $f(x)=3x+2$ and $g(x)=-(0.5)x+4$**

| $x$ | $f(x)=3x+2$ | $g(x)=-(0.5)x+4$ |
|---|---|---|
| -3 | -7 | 5.5 |
| -2.5 | -5.5 | 5.25 |
| -2 | -4 | 5 |
| -1.5 | -2.5 | 4.75 |
| -1 | -1 | 4.5 |
| -0.5 | 0.5 | 4.25 |
| 0 | 2 | 4 |
| 0.5 | 3.5 | 3.75 |
| 1 | 5 | 3.5 |
| 1.5 | 6.5 | 3.25 |
| 2 | 8 | 3 |
| 2.5 | 9.5 | 2.75 |
| 3 | 11 | 2.5 |
| 3.5 | 12.5 | 2.25 |
| 4 | 14 | 2 |
| 4.5 | 15.5 | 1.75 |

Using this table, students can reason that since $f(x) = 3.5$ at $x = 0.5$ and $f$ is increasing, and $g(x) = 3.5$ at $x = 1$ and $g$ is decreasing, the two functions must take on the same value somewhere between these values (**MP.3, MP.6**). In this example, since the original equation is of degree one, students know that there is only one solution, and using finer increments of $x$ will approximate the solution. Examining graphs and tables and solving equations algebraically help students to make connections between these various representations of functions and equations.

## Conceptual Category: Geometry

The standards for grades seven and eight introduced students to seeing two-dimensional shapes as part of a generic plane (the *Euclidean plane*) and exploring transformations of this plane as a way to determine whether two shapes are congruent or similar. These notions are formalized In Mathematics I, and students use transformations to prove geometric theorems about triangles. Students then apply these triangle congruence theorems to prove other geometric results, engaging throughout in standard **MP.3.**

### Congruence — G-CO

**Experiment with transformations in the plane.**

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

**Understand congruence in terms of rigid motions.** [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

**Make geometric constructions.** [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

In the Geometry conceptual category, the commonly held (but imprecise) definition that shapes are congruent when they "have the same size and shape" is replaced by a more mathematically precise one (**MP.6**): *Two shapes are congruent if there is a sequence of rigid motions in the plane that takes one shape exactly onto the other.* This definition is explored intuitively in the grade-eight standards, but it is investigated more closely in Mathematics I. In grades seven and eight, students experimented with transformations in the plane, but in Mathematics I they build more precise definitions for the *rigid motions* (rotation, reflection, and translation) based on previously defined and understood terms such as *angle, circle, perpendicular line, point, line, between,* and so forth (**G-CO.1, 3–4**). Students base their understanding of these definitions on their experience with transforming figures using patty paper, transparencies, or geometry software (**G-CO.2–3, 5; MP.5**), something they started doing in grade eight. These transformations should be investigated both in a general plane as well as on a coordinate system—especially when transformations are explicitly described by using precise names of points, translation vectors, and specific lines.

| **Example: Defining Rotations** | **G-CO.4** |
|---|---|

Mrs. B wants to help her class understand the following definition of a *rotation:*

> A *rotation* about a point $P$ through angle $\alpha$ is a transformation $A \mapsto A'$ such that (1) if point $A$ is different from $P$, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point $A$ is the same as point $P$, then $A' = A$.

Mrs. B gives her students a handout with several geometric shapes on it and a point, $P$, indicated on the page. In pairs, students copy the shapes onto a transparency sheet and rotate them through various angles about $P$; then they transfer the rotated shapes back onto the original page and measure various lengths and angles as indicated in the definition.

While justifying that the properties of the definition hold for the shapes given to them by Mrs. B, the students also make some observations about the effects of a rotation on the entire plane. For example:

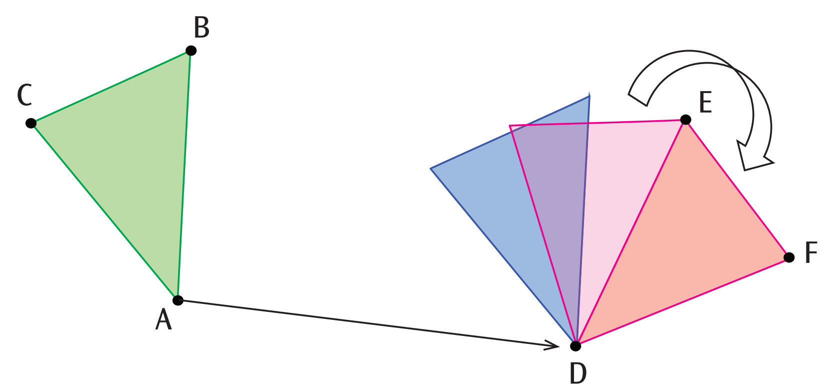
- Rotations preserve lengths.
- Rotations preserve angle measures.
- Rotations preserve parallelism.

In a subsequent exercise, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points, both part of the object and not part of the object.

In standards **G-CO.6–8**, geometric transformations are given a more prominent role in the higher mathematics geometry curriculum than perhaps ever before. The new definition of *congruence* in terms of rigid motions applies to any shape in the plane, whereas previously, congruence seemed to depend on criteria that were specific only to particular shapes. For example, the side–side–side (SSS) congruence criterion for triangles did not extend to quadrilaterals, which seemed to suggest that congruence was a notion dependent on the shape that was considered. Although it is true that there are specific criteria for determining congruence of certain shapes, the basic notion of congruence is the same for all shapes. In the CA CCSSM, the SSS criterion for triangle congruence is a *consequence* of the definition of congruence, just as the fact that if two polygons are congruent, then their sides and angles can be put into a correspondence such that each corresponding pair of sides and angles is congruent. This concept comprises the content of standards **G-CO.7** and **G-CO.8,** which derive congruence criteria for triangles from the new definition of congruence.

Further discussion of standards **G-CO.7** and **G-CO.8** is warranted here. Standard **G-CO.7** explicitly states that students show that two triangles are congruent *if and only if* corresponding pairs of sides and corresponding pairs of angles are congruent (**MP.3**). The depth of reasoning here is fairly substantial at this level, as students must be able to show, using rigid motions, that congruent triangles have congruent corresponding parts and that, conversely, if the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions that takes one triangle to the other. The second statement may be more difficult to justify than the first for most students, so a justification is presented here.

**Figure M1-3. Illustration of the Reasoning That Congruent Corresponding Parts Imply Triangle Congruence**




Point $A$ is translated to $D$, the resulting image of $\Delta ABC$ is rotated so as to place $B$ onto $E$, and the image is then reflected along line segment $DE$ to match point $C$ to $F$.

Suppose there are two triangles $\Delta ABC$ and $\Delta DEF$ such that the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$ results in pairs of sides and pairs of angles being congruent. If one triangle were drawn on a fixed piece of paper and the other drawn on a separate transparency, then a student could illustrate a translation, $T$, that takes point $A$ to point $D$. A simple rotation $R$ about point $A$, if necessary, takes point $B$ to point $E$, which is certain to occur because $\overline{AB} \cong \overline{DE}$ and rotations preserve lengths. A final step that may be needed is a reflection $S$ about the side $AB$, to take point $C$ to point $F$. It is important to note why the image of point $C$ is actually $F$. Since $\angle A$ is reflected about line $\overleftrightarrow{AB}$, its measure is preserved. Therefore, the image of side $\overline{AC}$ at least lies on line $\overline{DF}$, since $\angle A \cong \angle D$. But since $\overline{AC} \cong \overline{DF}$, it must be the case that the image of point $C$ coincides with $F$. The previous discussion amounts to the fact that the sequence of rigid motions, $T$, followed by $R$, followed by $S$, maps $\Delta ABC$ exactly onto $\Delta DEF$. Therefore, if it is known that the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions carrying one onto the other; that is, they are congruent. Figure M1-3 presents the steps in this reasoning.

Similar reasoning applies for standard **G-CO.8**, in which students justify the typical triangle congruence criteria such as ASA, SAS, and SSS. Experimentation with transformations of triangles where only two of the criteria are satisfied will result in counterexamples, and geometric constructions of triangles of prescribed side lengths (e.g., in the case of SSS) will leave little doubt that any triangle constructed with these side lengths will be congruent to another, and therefore that SSS holds (**MP.7**). Note that in standards **G-CO.1–8,** formal proof is not required. Students are asked to use transformations to show that particular results are true.

## Expressing Geometric Properties with Equations G-GPE

**Use coordinates to prove simple geometric theorems algebraically.** [Include distance formula; relate to Pythagorean Theorem.]

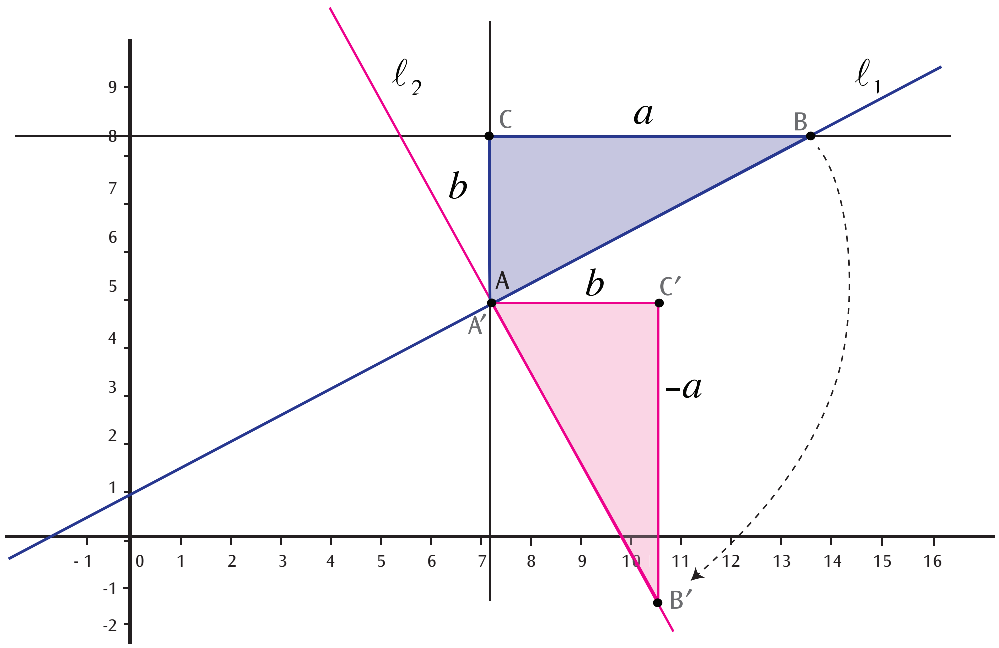
4. Use coordinates to prove simple geometric theorems algebraically.
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

The intersection of algebra and geometry is explored in this cluster of standards. Standard **G-GPE.4** calls for students to use coordinates to prove simple geometric theorems. For instance, they prove that a figure defined by four points is a rectangle by proving that lines containing opposite sides of the figure are parallel and lines containing adjacent sides are perpendicular. Students must be fluent in finding slopes and equations of lines (where necessary) and understand the relationships between the slopes of parallel and perpendicular lines (**G-GPE.5**).

Many simple geometric results can be proved algebraically, but two results of high importance are the slope criteria for parallel and perpendicular lines. Students in grade seven began to study lines and linear equations; in Mathematics I, they not only use relationships between slopes of parallel and perpendicular lines to solve problems, but they also justify why these relationships are true. An intuitive argument for why parallel lines have the same slope might read, "Since the two lines never meet, each line must *keep up* with the other as we travel along the slopes of the lines. So it seems obvious that their slopes must be equal." This intuitive thought leads to an equivalent statement: if given a pair of linear equations $\ell_1 : y = m_1 x + b_1$ and $\ell_2 : y = m_2 x + b_2$ (for $m_1, m_2 \neq 0$) such that $m_1 \neq m_2$—that is, such that *their slopes are different*—then the lines must intersect. Solving for the intersection of the two lines yields the $x$-coordinate of their intersection to be $x = \frac{b_2 - b_1}{m_1 - m_2}$, which surely exists because $m_1 \neq m_2$. It is important for students to understand the steps of the argument and comprehend why proving this statement is equivalent to proving the statement "If $\ell_1 \parallel \ell_2$, then $m_1 = m_2$" (**MP.1, MP.2**).

In addition, students are expected to justify why the slopes of two non-vertical perpendicular lines $\ell_1$ and $\ell_2$ satisfy the relationship $m_1 = -\frac{1}{m_2}$, or $m_1 \bullet m_2 = -1$. Although there are numerous ways to do this, only one way is presented here, and dynamic geometry software can be used to illustrate it well (**MP.4**). Let $\ell_1$ and $\ell_2$ be any two non-vertical perpendicular lines. Let $A$ be the intersection of the two lines, and let $B$ be any other point on $\ell_1$ above $A$. A vertical line is drawn through $A$, a horizontal line is drawn through $B$, and $C$ is the intersection of those two lines. $\Delta ABC$ is a right triangle. If $a$ is the horizontal displacement $\Delta x$ from $C$ to $B$, and $b$ is the length of $\overline{AC}$, then the slope of $\ell_1$ is $m_1 = \frac{\Delta y}{\Delta x} = \frac{b}{a}$. By rotating $\Delta ABC$ clockwise around $A$ by 90 degrees, the hypotenuse $\overline{AB'}$ of the rotated triangle $\Delta AB'C'$ lies on $\ell_2$. Using the legs of $\Delta AB'C'$, students see that the slope of $\ell_2$ is $m_2 = \frac{\Delta y}{\Delta x} = \frac{-a}{b}$. Thus $m_1 \bullet m_2 = \frac{b}{a} \bullet \frac{-a}{b} = -1$. Figure M1-4 illustrates this proof (**MP.1, MP.7**).

**Figure M1-4. Illustration of the Proof That the Slopes of Two Perpendicular Lines Are Opposite Reciprocals of One Another**




The proofs described above make use of several ideas that students learned in Mathematics I and prior courses—for example, the relationship between equations and their graphs in the plane (**A-REI.10**) and solving equations with variable coefficients (**A-REI.3**). An investigative approach that first uses several examples of lines that are perpendicular and their equations to find points, construct triangles, and decide if the triangles formed are right triangles will help students ramp up to the second proof (**MP.8**). Once more, the reasoning required to make sense of such a proof and to communicate the essence of the proof to a peer is an important goal of geometry instruction (**MP.3**).

## Conceptual Category: Statistics and Probability

In Mathematics I, students build on their understanding of key ideas for describing distributions—shape, center, and spread—presented in the standards for grades six through eight. This enhanced understanding allows students to give more precise answers to deeper questions, often involving comparisons of data sets.

## Interpreting Categorical and Quantitative Data — S-ID

### Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

**Summarize, represent, and interpret data on two categorical and quantitative variables.** [Linear focus; discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

### Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★

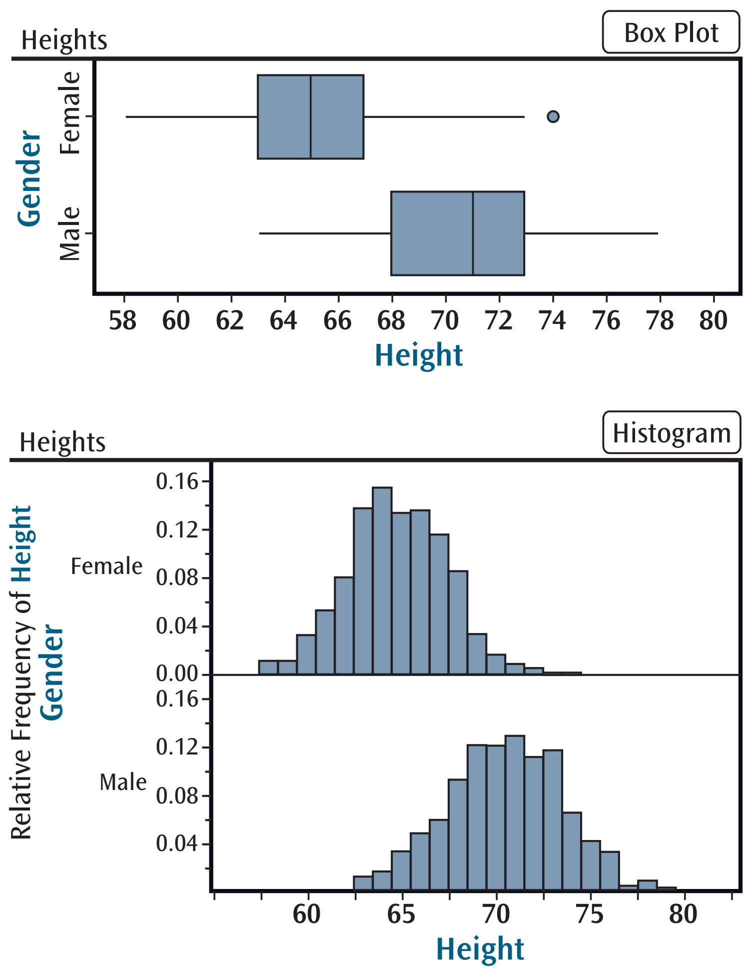
Standards **S-ID.1–6** support standards **S-ID.7–9**, in the sense that the former standards extend concepts students began to learn in grades six through eight. Students use the shape of the distribution and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. Students may use parallel box plots or histograms to compare differences in the shape, center, and spread of comparable data sets (**S-ID.1–2**).

| Example | S-ID.2 |
|---|---|

The following graphs show two ways of comparing height data for males and females in the 20–29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called parallel (or side-by-side) box plots and parallel histograms (**S-ID.1**). The parallel box plots show an obvious difference in the medians and the interquartile ranges (IQRs) for the two groups; the medians for males and females are, respectively, 71 inches and 65 inches, while the IQRs are 5 inches and 4 inches. Thus, male heights center at a higher value but are slightly more variable.

The parallel histograms show the distributions of heights to be mound shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer questions about it solely from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean (**S-ID.2–3**). Students also observe that the two measures of center—median and mean—tend to be close to each other for symmetric distributions.

**Comparing heights of males and females**




Heights of U.S. males and females in the 20–29 age group

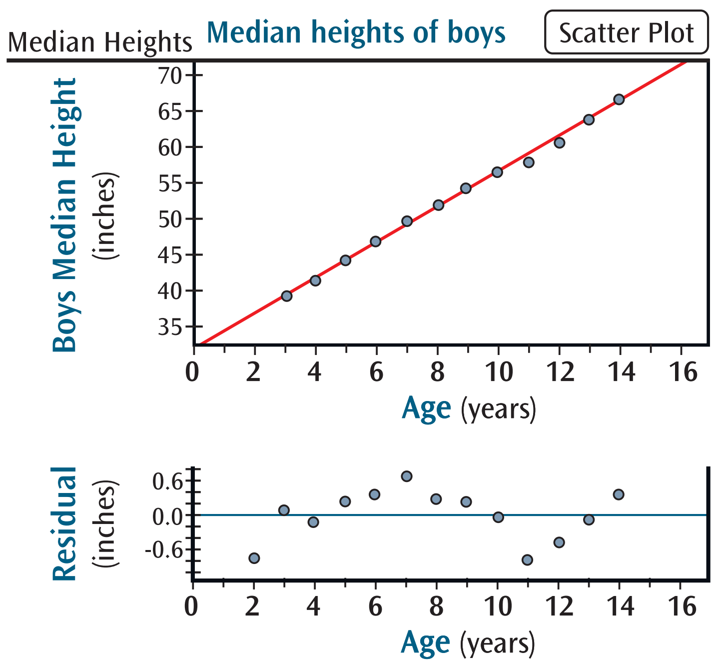
*Source:* United States Census Bureau 2009 (Statistical Abstract of the United States, Table 201).

Adapted from UA Progressions Documents 2012d, 3.

Students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data (**S-ID.5–6**). Students have seen scatter plots in the grade-eight standards and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model indicates about the relationship. Students must learn to take a careful look at scatter plots, as sometimes the "obvious" pattern does not tell the whole story and may be misleading. A line of best fit may appear to fit data almost perfectly, while an examination of the *residuals*—the collection of differences between corresponding coordinates on a least squares line and the actual data value for a variable—may reveal more about the behavior of the data.

**Example** **S-ID.6b**

Students must learn to look carefully at scatter plots, as sometimes the "obvious" pattern may not tell the whole story and could even be misleading. The graphs below show the median heights of growing boys from the ages of 2 through 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit (**S-ID.6c**). However, the residuals—the differences between the corresponding coordinates on the least squares line and the actual data values for each age—reveal additional information. A plot of the residuals shows that growth does not proceed at a constant rate over those years.




Boys Median Height = 31.6 in + (2.47 in/yr) Age; $r^2$=1.00

*Source:* Centers for Disease Control and Prevention (CDC) 2002.

Adapted from UA Progressions Documents 2012d, 5.

Finally, students extend their work from topics covered in the grade-eight standards and other topics in Mathematics I to interpret the parameters of a linear model in the context of data that it represents. They compute *correlation coefficients* using technology and interpret the value of the coefficient (**MP.4, MP.5**). Students see situations where correlation and causation are mistakenly interchanged, and they are careful to closely examine the story that data and computed statistics try to tell (**S-ID.7–9**).

# Mathematics I Overview

## Number and Quantity

### Quantities

- Reason quantitatively and use units to solve problems.

## Algebra

### Seeing Structure in Expressions

- Interpret the structure of expressions.

### Creating Equations

- Create equations that describe numbers or relationships.

### Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

## Functions

### Interpreting Functions

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

### Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

### Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

**Mathematical Practices**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

# Mathematics I Overview *(continued)*

## Geometry

### Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Make geometric constructions.

### Expressing Geometric Properties with Equations

- Use coordinates to prove simple geometric theorems algebraically.

## Statistics and Probability

### Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

## Number and Quantity

### Quantities N-Q

**Reason quantitatively and use units to solve problems.** [Foundation for work with expressions, equations, and functions]

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
2. Define appropriate quantities for the purpose of descriptive modeling. ★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

## Algebra

### Seeing Structure in Expressions A-SSE

**Interpret the structure of expressions.** [Linear expressions and exponential expressions with integer exponents]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of $P$ and a factor not depending on $P$.* ★

### Creating Equations A-CED

**Create equations that describe numbers or relationships.** [Linear and exponential (integer inputs only); for **A-CED.3**, linear only]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance $R$.* ★

# M1 Mathematics I

## Reasoning with Equations and Inequalities A-REI

**Understand solving equations as a process of reasoning and explain the reasoning.** [Master linear; learn as general principle.]

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

**Solve equations and inequalities in one variable.**

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [Linear inequalities; literal equations that are linear in the variables being solved for; exponential of a form, such as $2^x = 1/16$.]

**3.1 Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context. CA**

**Solve systems of equations.** [Linear systems]

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

**Represent and solve equations and inequalities graphically.** [Linear and exponential; learn as general principle.]

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
11. Explain why the $x$-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

# Functions

## Interpreting Functions F-IF

**Understand the concept of a function and use function notation.** [Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences.]

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If $f$ is a function and $x$ is an element of its domain, then $f(x)$ denotes the output of $f$ corresponding to the input $x$. The graph of $f$ is the graph of the equation $y = f(x)$.

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

**Interpret functions that arise in applications in terms of the context.** [Linear and exponential (linear domain)]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

**Analyze functions using different representations.** [Linear and exponential]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

| Building Functions | F-BF |
|---|---|

**Build a function that models a relationship between two quantities.** [For F-BF.1, 2, linear and exponential (integer inputs)]

1. Write a function that describes a relationship between two quantities. ★
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★

**Build new functions from existing functions.** [Linear and exponential; focus on vertical translations for exponential.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

## Linear, Quadratic, and Exponential Models — F-LE

**Construct and compare linear, quadratic, and exponential models and solve problems.** [Linear and exponential]

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. ★
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★

**Interpret expressions for functions in terms of the situation they model.** [Linear and exponential of form $f(x)=b^x+k$]

5. Interpret the parameters in a linear or exponential function in terms of a context. ★

# Geometry

## Congruence — G-CO

**Experiment with transformations in the plane.**

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

**Understand congruence in terms of rigid motions.** [Build on rigid motions as a familiar starting point for development of concept of geometric proof.]

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

**Make geometric constructions.** [Formalize and explain processes.]

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

### Expressing Geometric Properties with Equations — G-GPE

**Use coordinates to prove simple geometric theorems algebraically.** [Include distance formula; relate to Pythagorean Theorem.]

4. Use coordinates to prove simple geometric theorems algebraically.
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★

## Statistics and Probability

### Interpreting Categorical and Quantitative Data — S-ID

**Summarize, represent, and interpret data on a single count or measurement variable.**

1. Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

**Summarize, represent, and interpret data on two categorical and quantitative variables.** [Linear focus; discuss general principle.]

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★
   b. Informally assess the fit of a function by plotting and analyzing residuals. ★
   c. Fit a linear function for a scatter plot that suggests a linear association. ★

**Interpret linear models.**

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
9. Distinguish between correlation and causation. ★