# Preschool Through Third Grade (P–3) Learning Progressions

## Mathematics

California Department of Education

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# Introduction

The *Preschool Through Third Grade (P–3) Learning Progressions in Mathematics* shows the correspondence between the *Preschool/ Transitional Kindergarten Learning Foundations (PTKLF) in Mathematics* and the *California Common Core State Standards: Mathematics (CCSS Mathematics)*. This resource includes the knowledge and skills children develop over the P–3 continuum and in-practice examples that illustrate how educators can use playful, inquiry-based teaching practices to engage children in developmentally appropriate, culturally sustaining mathematics learning experiences.

## How Does Children's Early Development in Mathematics Inform P–3 Teaching and Learning?

Children's early mathematics skills provide the foundation for mathematics learning in the early elementary years. Children are born with the motivation and ability to explore mathematical concepts.[1] In early childhood, children can develop strong foundations for learning core areas in mathematics such as number and counting, operations and algebraic thinking, measurement and data, and geometry and spatial thinking. For example, from preschool through third grade, children's understanding of number progresses from understanding the meaning of counting and how to count small sets of objects in the preschool years to understanding the number system beyond 100 and deepening understanding of number relationships in the early elementary years. It is important for educators supporting children in preschool through third grade to remember that:

- Children are curious about the world and learn mathematics through authentic, meaningful experiences.[2] How children engage in exploring mathematical concepts in their everyday lives varies based on their experiences as individuals within their family and community. Creating open-ended tasks with opportunities to use various approaches and discover different solutions allows children to experiment with mathematical concepts and develop deeper understandings.
- Children's feelings toward mathematics impact their learning.[3] Creating engaging and positive mathematics experiences and interacting with each and every child as capable of learning mathematics supports their desire to learn mathematics and develop positive identities as mathematics learners.
- Children bring with them a wealth of knowledge and skills in mathematics—learned in formal education settings and also learned informally through daily experiences in children's homes and communities. Educators can leverage this knowledge to build their understanding and use of mathematical concepts.
- Learning about children's home lives and cultures can help educators build meaningful connections to children's lives during learning experiences.
- Children who use more than one language learn and communicate about mathematics concepts in multiple

languages. Supporting the home language can help children draw on all of their mathematical skills and knowledge across languages.[4]

Furthermore, development is dynamic. Children draw on other areas of development to support them to solve mathematical problems. Similarly, solving mathematical problems supports the development of knowledge and skills in other domains. For example, children use and develop approaches to learning skills, including persevering, problem-solving, and collaborating with others, when solving mathematical problems.[5] Executive function skills support children to hold and manipulate mathematical information in their minds as they solve problems, use information flexibly to solve problems in different ways, and attend to relevant information while inhibiting distractions when solving mathematics problems.[6] Language and mathematics are also strongly related.[7] For example, children use language skills when solving word problems, explaining their reasoning, and constructing mathematical arguments. It is important to consider the whole child to understand how development across domains may affect and support early mathematics learning.

# P–3 Mathematics Teaching Practices

- **Use observations of children's mathematics skills and knowledge to support and challenge children.** Observe and notice individual children's emerging mathematics skills and understanding to build on their discoveries, strengths, and interests. Learn about the various ways children may express their mathematical knowledge and skills. Pay attention to their ideas to identify everyday moments where mathematical concepts can be clarified, extended, or reinforced.[8]
- **Foster feelings of belonging for all children by providing learning opportunities and materials that reflect and are responsive to children's daily lives, abilities, cultures, and languages.**[9] Connect mathematics to children's everyday lived experiences, their homes, and cultures. On a regular basis, invite children to use mathematics to solve problems that are meaningful to them and that connect to their daily lives, existing knowledge, and cultural and linguistic knowledge and experiences. Encourage peer collaboration by creating opportunities for all children, including children with disabilities, to work together to solve mathematical problems.
- **Encourage children to apply and develop their understanding of mathematics through active investigation of meaningful questions or problems.**[10] Provide children with opportunities to engage in meaningful problem-solving and mathematical reasoning through hands-on, open-ended, playful learning activities. Generate interest by posing questions and prompts that require children to apply their mathematics knowledge and skills to scenarios and questions of interest.
- **Provide a mathematically rich learning environment by offering a variety of objects and tools for children to explore and make meaning of mathematical concepts.**[11] Young children benefit from engaging in a variety of learning experiences using concrete objects to explore and make meaning of mathematical concepts in a variety of contexts or tasks. This may include various visual references, like number lines, posters, tables, and graphs; measuring tools, including standard and nonstandard units; and a variety of items that children can use to count, sort, compare, and compute with, like counting chips, beads, shells, pine cones, paper cups, and spoons. The outdoors also offer a mathematically rich environment where children can engage in mathematical thinking while interacting with the world around them. Educators can modify materials, if

needed, to support all children, including children with disabilities, to fully engage in mathematics learning experiences.

- **Facilitate mathematics language and discourse.**[12] Create opportunities for children to use mathematical language throughout the day, in English, the home language, varieties of English such as African American Vernacular English, and informal mathematics language. If children use augmentative and alternative communication devices or visuals to communicate, ensure mathematical language is available on devices and visual cues. Encourage children to ask questions, describe, compare, explain, record, share, and critique their own solutions. Invite children to grapple with mathematics concepts and share their ideas with each other.
- **Encourage children to explore and express mathematics ideas through multiple representations.** Provide opportunities for children to use various representations of mathematics ideas, including drawings, visuals, models, tables, graphs, manipulatives, 3D models, or through movement. Using different representations to express mathematics ideas deepens children's understanding and helps engage all learners.[13] Modify learning experiences that offer multiple representations of mathematics ideas to meet children's diverse abilities and preferences.

# How Are the P–3 Learning Progressions in Mathematics Organized?

## Key Areas

*The P–3 Learning Progressions in Mathematics* delineate the development of children's mathematics knowledge and skills from preschool through third grade in six key areas:

- **Key Area 1: Counting and Cardinality**
  Knowing numbers in sequence and understanding the relation between number and quantity. Learning progressions include Count in Sequence, Connect Counting to Cardinality, Read and Write Numerals, and Compare Numbers.
- **Key Area 2: Operations and Algebraic Thinking**
  Solving arithmetic problems and reasoning about ideas such as adding and subtracting, number composition, and patterns. Learning progressions include Represent and Solve Addition and Subtraction Problems, Compose and Decompose Numbers, Add and Subtract Within a Range (5, 10, 20, 100, 1000), and Identify, Create, and Reason with Patterns.
- **Key Area 3: Number and Operations in Base Ten**
  Using the base-ten system to extend the count sequence, understand place value, and solve problems using different operations. Learning progressions include Understand Place Value and Use Place Value to Add and Subtract.
- **Key Area 4: Measurement**
  Comparing, ordering, and measuring objects by a variety of attributes, such as length, weight, or capacity. Learning progressions include Describe, Compare, Order Measurable Attributes, and Measure and Estimate Length.
- **Key Area 5: Data**
  Classifying data in different ways, representing data through various means, and interpreting data to make meaning. Learning progressions include Classify, Represent, and Interpret Data.
- **Key Area 6: Geometry and Spatial Thinking**
  Identifying shapes, describing shapes and their attributes, creating different shapes, and reasoning with shapes. Learning progressions include Identify, Describe, and Reason with Shapes and Their Attributes and Create, Compose, and Partition Shapes.

## Learning Progressions Tables

Each key area includes learning progressions tables that present children's development of skills and concepts from preschool through third grade drawn from the *PTKLF in Mathematics* and the *CCSS Mathematics*. The progressions include the foundations and standards that are most conceptually aligned to illustrate a continuum of how children develop knowledge and skills in each area. As such, not all the mathematical foundations (*PTKLF*) or standards (*CCSS Mathematics*) are included in the progressions.

Each key area learning progression table also includes a relevant Content Connection, as described in the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (*Mathematics Framework*). The Content Connections define overarching mathematical concepts that create coherence among

standards and across grades. The inclusion of Content Connections further unifies the foundations and standards included in the key area learning progression tables. For more information on Content Connections, refer to the *Mathematics Framework*, Chapter 6: Investigating and Connecting, Transitional Kindergarten through Grade Five.

## In-Practice Examples

After the learning progressions tables in each key area, there are in-practice examples that illustrate playful, inquiry-based, and culturally sustaining practices that educators can use to support children of diverse abilities and backgrounds in mathematics learning across different grades. A section, "Highlights from the In-Practice Examples," at the end of the in-practice examples in each key area offers a description of how educators in the examples used teaching practices to support learning and embrace the diversity of learners in their classrooms.

At the end of the document there are appendices that provide further information to support teaching. Appendix A offers guidance on strategies educators can use to support English language development (ELD) for multilingual learners. Appendix B includes references to vignettes in other resources to help further illustrate the teaching practices included in the in-practice examples.

Additionally, included in this document is a description of how children from preschool through third grade use mathematical practices to engage meaningfully in the key areas outlined above. Information related to mathematical practices is adapted from the *California Common Core State Standards for Mathematics Standards for Mathematical Practices* and includes the following eight practices: make sense of problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments and critique the reasoning of others, model with mathematics, use appropriate tools strategically, attend to precision, look for and make use of structure, and look for and express regularity in repeated reasoning.

# Standards for Mathematical Practice

*California Common Core State Standards for Mathematics Standards for Mathematical Practice* (Mathematical Practices) describe the skills that allow children to develop a deep understanding of mathematics, including the concepts outlined within the six key area progression tables. Below is an outline of the eight practices and descriptions of how children from preschool into the early elementary years use these practices to develop an understanding of mathematical concepts.

## Mathematical Practices

### Preschool Through Early Elementary

**Make sense of problems and persevere in solving them**
Children in preschool/transitional kindergarten (TK) apply mathematical thinking to solve problems in everyday activities and interactions. They persist and try a variety of ways to solve problems. In the early elementary years, children continue to develop their skills. They persevere through more complex problems with more advanced, multi-step solutions. Older children might also engage in more strategic planning and more frequent review and evaluation of their solutions.

**Reason abstractly and quantitatively**
Children in preschool/TK reason about abstract mathematical ideas such as quantity and equality using concrete objects (for example, "I have two cars and you have two cars. We have four cars"). As children progress into the early elementary years, they might relate to mathematical problems in more symbolic ways (for example, using symbolic representations: 2 + 2 = 4).

**Construct viable arguments and critique the reasoning of others**
Children in preschool/TK construct arguments about solutions, explain their reasoning, and think critically about the solutions of others. As children progress into the early elementary years, they develop more accurate explanations and more viable arguments. Children are also better able to understand the diverse perspectives of others, allowing them to benefit more from critiquing approaches with others.

**Model with mathematics**
Children in preschool/TK use models such as drawings, constructions, or their bodies to illustrate their thinking and solve mathematical problems. As children progress into the early elementary years, they use a wider range of models, including written equations, graphs, and tables.

**Use appropriate tools strategically**
Children in preschool/TK use a variety of tools to solve mathematical problems (for example, concrete objects to solve addition or subtraction problems or a piece of string, a ruler, or a scale for measuring or ordering objects). As children progress into the early elementary years, they might use a greater variety of tools (for example, base-ten blocks, ten frames, and standard units of measure), with greater precision, to solve mathematical problems.

**Attend to precision**
Children in preschool/TK begin to apply mathematical skills (for example, counting, comparing, adding, and subtracting), represent mathematical ideas, and communicate their mathematical understanding with clarity and attention to precision. As children progress into the early elementary years, and gain knowledge and experience, they communicate, represent mathematical ideas, and apply mathematical skills with increased clarity and attention to precision.

**Look for and make use of structure**
Children in preschool/TK recognize structures and rules in mathematics (for example, noticing that the six on a die is represented with three and three dots or two and two and two dots). As children progress into the early elementary years, they continue to look for and make use of structures (for example, they expand their understanding of the number system using the base-ten structure). They build on these skills by using more complex rules and structures (for example, the commutative property of addition) to solve a variety of problems in various contexts.

**Look for and express regularity in repeated reasoning**
Children in preschool notice patterns and regularity in mathematics (for example, a whole number plus one is the next number in the number list, triangles have three sides). As children progress into the early elementary years, they continue to recognize and apply patterns in a variety of mathematical areas (for example, recognizing patterns within the base-ten system as they work with larger numbers, or recognizing that multiplication is repeated addition when using more advanced operations).

Educators can support children in developing the skills described in the Mathematical Practices during everyday activities and through learning experiences provided in an evidence-based mathematics curriculum. See the section "Teaching Practices" to learn more about how educators can provide a rich learning environment that allows children to apply their reasoning and problem-solving skills in meaningful mathematical explorations.

# P–3 Learning Progressions in Mathematics

## Key Area 1: Counting and Cardinality

**Content Connections: Exploring Changing Quantities**


From a young age, children begin paying attention to quantity and use number words to count and label quantities. From preschool through the early elementary years, they expand their knowledge of counting and how numbers in the count list represent quantities (an understanding of cardinality). During the early elementary years, children become more fluent and accurate in counting. They extend their understanding of the number system beyond 100 and deepen their understanding of number relationships. Children apply this understanding of numbers to several other areas of mathematics, including operations and algebraic thinking, interpreting data, and geometry.

## Learning Progression Table 1.1: Count in Sequence

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.1.1. Recite numbers in order from one to ten or higher with no more than a few errors. | PTK.Later.1.1. Recite numbers in order from one to thirty with no more than a few errors. Count forward from a number other than one. | K.CC.1. Count to 100 by ones and by tens.<br><br>K.CC.2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1). | 1.NBT.1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.* | 2.NBT.2. Count within 1000; skip-count by 2s, 5s, 10s, and 100s. | *Children in third grade continue to build on their understanding of the number system to count in sequence or skip count, advancing to numbers beyond 1000.* |

* This standard is represented within multiple concepts in Key Area 1 because it includes three important skills: (1) counting in sequence, described as, "Count to 120, starting at any number less than 120," (2) reading and writing numerals, described as, "read and write numerals," and (3) cardinality, described as, "represent a number of objects with a written numeral."

## Learning Progression Table 1.2: Connect Counting to Cardinality

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.1.2. Count five objects or more using one-to-one correspondence (one object for each number word).<br><br>PTK.Early.1.3. Answer the question "How many ... ?" May repeat the last number word in the number list after counting but is still developing an understanding that the number name of the last object counted represents the total number of objects in the group. | PTK.Later.1.2. Count ten objects or more using one-to-one correspondence (one object for each number word).<br><br>PTK.Later.1.3. Consistently demonstrate understanding, when counting, that the number name of the last object counted represents the total number of objects in the group. | K.CC.4. Understand the relationship between numbers and quantities; connect counting to cardinality.<br><br>K.CC.5. Count to answer "how many?" questions for about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects. | 1.NBT.1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.*<br><br>*Counting with understanding is foundational to early elementary mathematics learning in all key areas (for example, in solving addition and subtraction problems:* 1.OA.5. Relate counting to addition and subtraction [for example, by counting on 2 to add 2]). | *Counting with understanding is foundational to early elementary mathematics learning in all key areas. (for example, in reasoning about shapes:* 2.G.2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.) | *Counting with understanding is foundational to early elementary mathematics learning in all key areas (for example, in geometric measurement:* 3.G.6. Measure areas by counting unit squares [square cm, square m, square in, square ft, and improvised units]). |

## Learning Progression Table 1.3: Read and Write Numerals

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.1.5. Recognize and name a few written numerals under 10 | PTK.Later.1.5. Recognize and name all written numerals through 10. | K.CC.3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects). | 1.NBT.1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.* | 2.NBT.3. Read and write numbers up to 1000 using base-ten numerals, number names, and expanded form. | *Children in third grade continue to build on reading and writing numbers, advancing to larger numbers.* |

* This standard is represented within multiple concepts in Key Area 1 because it includes three important skills: (1) counting in sequence, described as, "Count to 120, starting at any number less than 120," (2) reading and writing numerals, described as, "read and write numerals," and (3) cardinality, described as, "represent a number of objects with a written numeral."

## Learning Progression Table 1.4: Compare Numbers

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.1.6. Compare visually (with or without counting) two groups of objects that are clearly equal or different in size and communicate, "same" or "more." | PTK.Later.1.6. Compare two groups of objects by counting and communicating, "more," "same," "less," or "fewer." | K.CC.6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, for example, by using matching and counting strategies.<br><br>K.CC.7. Compare two numbers between 1 and 10 presented as written numerals. | 1.NBT.3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <. | 2.NBT.4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons. | *Children in third grade move from focusing on comparing whole numbers to also include comparing fractions.*<br><br>3.NF.3. Explain equivalence of fractions in special cases and compare fractions by reasoning about their size (see standards 3.NF.3 a–d.). |

# In-Practice Examples

## Learning Progression 1.4: Compare Numbers

### Blue Corn or Okra: What Should We Plant?

The in-practice examples below demonstrate how educators support children in preschool and first grade to compare numbers (refer to the learning progression Compare Numbers) as a way to answer a meaningful question—should we plant blue corn or okra seeds in our garden? While the in-practice examples below are grade specific, educators in other grades can adapt similar strategies to support children to compare numbers.

### As you read,

Notice how children of different ages and in different grades:

- Develop increasingly complex number comparison skills.
- Use various strategies to compare numbers.

Notice how educators:

- Facilitate children's engagement in meaningful mathematics investigations and problem-solving.
- Use observations of children's mathematics skills and knowledge to support and challenge their learning.
- Provide a mathematically rich environment and create opportunities for exploration for all children.
- Create opportunities for children to use mathematics language and engage in mathematics discourse in English and the home language.
- Foster children's sense of belonging by offering experiences that connect to children's families and cultural traditions.
- Make modifications to the environment, materials, and/or instruction to ensure each and every child, including children with disabilities, can participate.

## Which Group Has More? (Preschool)

Ms. Greene's preschool class (children ages three and four years old) is getting ready for more planting in their garden. In preparation for planting, Ms. Greene talks with families to learn more about the foods children might have in their homes. After selecting two popular food items that children are familiar with—blue corn and okra—Ms. Greene decides to let the children vote to determine which one to plant. "Today we are going to decide what type of plants we should add to our garden. I have blue corn seeds and okra seeds," says Ms. Greene as she points to photos of these plants, and she hands out blue corn and okra seeds for the children to observe. "Have any of you ever eaten blue corn or okra?" Ms. Greene asks, knowing that these are both foods children in her classroom have eaten at home.

"I eat okra! I love it when my Nana cooks भिंडी [okra, in Hindi]!" Tavish shares.

"Me too! My Gram cooks the fried okra!" adds J'Naya.

"I had blue corn mush for breakfast!" said Nizhoni.

After the children share their experiences with blue corn and okra, they decide they can use voting to help them choose which seeds to plant.

Each child is given a popsicle stick to use as a concrete object to represent their vote for which seeds should be planted in their garden. They place their stick in one of two clear containers with a photo of either blue corn or okra on it. Once the voting is complete, Ms. Greene asks the class, "Which one has more votes? Turn and tell your partner what you think and how you figured it out." Ethan, a child with a visual impairment, feels the sticks in both containers before turning to his partner.

After the partners have shared their thinking with one another, Ms. Greene invites the children to share with the larger group their thoughts on which plant received more votes. Tavish points to the blue corn container and says, "This one!"

Ms. Greene asks, "How do you know the blue corn has more votes?"

Tavish quickly responds, "It's fuller!"

"You think the blue corn has more votes because the container looks fuller. What other ways might we figure out which one has more votes?" Ms. Greene asks.

**Educators can invite all children to express their mathematics knowledge and skills in a variety of ways, including gesturing, writing, drawing, and using their home languages.** Encouraging children to express themselves in any language allows them to draw on all their mathematical skills and knowledge across languages.

"I know, let's line them up!" says Grace. Ms. Greene hands Grace the blue corn sticks and gives the okra sticks to Tavish. Grace takes the sticks out of the containers and starts lining up the sticks in the blue corn group while Tavish works on lining up the sticks in the okra group. Grace points to the line of blue corn sticks and says, "This one has more because it's longer."

Ms. Greene nods, acknowledging Grace's response, and says, "So you noticed that the blue corn group is longer, which means it has more sticks than the okra group. Any other ideas?"

Isla says, "Let's count!"

"Great idea, Isla!" says Ms. Greene. Knowing that Isla sometimes counts in Spanish she asks, "Would you like to count in Spanish or in English?"

"Spanish!" Isla responds and begins to count the sticks in the blue corn group in her home language, Spanish, and then switches to English, "*Uno, dos, tres, cuatro, cinco, seis, siete, ocho*. The blue corn sticks has eight!" (One, two, three, four, five, six, seven, eight. The blue corn sticks has eight!)

"There are eight votes for blue corn!" says Ms. Greene.

Next, Ethan counts the okra votes. Isla gathers all of the sticks from the okra group and hands them to Ethan. He holds the bundle of sticks and counts as he places them back into the container one at a time. He determines that there are six okra sticks.

Ms. Green says, "How about if we all count both sets of votes together to make sure we didn't miss any? Isla, can you lead us in counting in Spanish first?" After the class counts both sets of votes in Spanish and in English, Isla jumps up and says, "Blue corn wins! Eight is bigger than six!"

Ms. Greene nods and says, "What does everyone think? Is eight more than six?" The children nod and say yes enthusiastically.

## Which is more, 32 or 28? (First Grade)

The children in Ms. King's class are excited to begin working on their garden to grow some of the vegetables that they like to eat at home—information the educator learned through partnering with families. After collecting votes from all the first-grade classrooms in the school to decide whether to plant blue corn or okra (foods many children have in their homes), the children in Ms. King's class work together to count the votes. The children count 32 votes for blue corn and 28 votes for okra. Ms. King writes 32 and 28 on the

board. “Let’s compare these numbers. Are there more blue corn or okra votes?” she asks. “Think to yourself and then tell your partner. For example, you might say, ‘There are more ____ than ____. I know this because ____.’ Make sure to listen carefully to your partner.”

After the children practice saying the phrase with Ms. King, they take turns sharing their ideas with their partners. Ms. King then calls on a few children randomly to share their thoughts. “There are more blue corn votes than okra votes!” shares Li.

“How do you know there are more blue corn votes?” Ms. King asks.

“Because 32 comes after 28,” Li answers and then starts counting forward, “28, 29, 30, 31, 32.”

“What do others think?” Ms. King asks.

“It’s blue corn because 30 is bigger,” Daniel responds.

Ms. King notices that Daniel is breaking the numbers into tens, the early stages of place value. To help encourage thinking related to place value, Ms. King gives the children counting chips. She says, “Remember we created groups of ten the other day? Let’s try that strategy to compare the number of votes.” The children work in two teams to put their counting chips in groups of tens and remaining ones. Li and Chen both speak Mandarin at home. Ms. King invites Li to work with Chen, who is new to the English language, so they can discuss mathematics using the language they choose. Ms. King observes children working and provides support as needed.

“The blue corn group has three groups of 10 and two leftover chips,” shares one group.

“Our okra group has two groups of 10 and eight chips left over,” shares the other group.

“So which group has more votes?” the educator asks.

Li shouts out, “Blue corn! ’Cause it has more 10s!”

## Highlights from the in-practice examples

**The educators support children to compare numbers in the context of a meaningful and authentic investigation—** deciding which seeds to plant in the class garden.* The investigation is also culturally sustaining because it includes food items that are a part of children's lives and cultures (for example, by partnering with families, the educators learn that many of the children have eaten blue corn or okra at home).

Here are some other examples of authentic problem-based investigations that encourage children to compare numbers.

- After going for a walk to collect rocks from the neighborhood, children sort their collection into a group of smooth rocks and a group of bumpy rocks. They wonder which type of rocks there are more of.
- Having logged the books children have read together in class each month, children compare the number of books they have read from month to month.
- After documenting the temperature outside each day, the children figure out if they experienced more warm days (above 80 degrees) or cooler days (below 79 degrees).

**The educators use observations of children's mathematics skills and knowledge to support and challenge their learning.** For example, in the first-grade classroom, Ms. King notices that Daniel is applying understanding of place value when breaking numbers into tens. Ms. King then builds on this understanding by reminding children of a strategy they have used recently. Ms. King provides counting chips to support children to separate the tens and ones as a strategy to compare numbers.

Here are some additional ideas related to observing how children compare numbers:

- As children compare numbers, notice what they are doing as well as what they are saying. Are they decomposing numbers in a certain way? Are they relying on their knowledge of the count sequence (for example, understanding which number comes first) or using a number line to support them?
- In addition to intentional mathematical lessons and experiences, observe children throughout the day to

* The use of gardens in schools can have many benefits for children's health and learning. When using food in lessons, it is important that educators are sensitive to children's personal experiences with, and access to, food—using food items in ways that are intentional.

identify moments when children authentically compare numbers.

**The educators create mathematically rich environments that support children to compare numbers.** They provide concrete objects to represent the children's votes and use principles of Universal Design for Learning (UDL) to encourage all children, including children with disabilities and a range of learning needs, to fully engage in mathematics. Providing children with popsicle sticks and counting chips allows them to physically group and compare objects in different ways. This strategy has added significance for a child with visual impairment who benefits from the tactile nature of the objects.

Here are some additional ideas related to the effective use of materials and the environment that encourages children to compare numbers.

- In addition to counting chips, children can use various everyday objects that are safe and appropriate for them, including culturally unique items from home to support counting and comparing. Examples include pom-pom balls, rocks, plastic spoons, beads, marbles, and dominoes.
- Educators can also expose children to various ways to represent their counts and facilitate comparisons, including pictographs, tally marks, ten frames, rekenreks, and bar graphs.

**The educators ask questions and probe children's ideas, engaging them in discourse focused on comparing numbers—in English and the home language.** The educators ask, "How do you know this one has more votes?" and "What other ways might we figure out which one has more votes?" The educators invite children to share their unique strategies for solving a comparison problem, allowing individual learners to participate at their level of understanding, use multiple representations, and communicate in English or their home language. In the first-grade classroom, the educator intentionally groups two children together that share the same home language. This approach allows the children to choose which language they prefer when discussing their mathematics learning.

Here are some other examples of questions and prompts to encourage children to compare numbers. Support children to express their ideas in ways that are most effective for them (for example, using their home language, gestures, or assistive technology).

- "How might we find out which one has more?"
- "Why do you think _____ is more than _____?"
- "What are some other ways we could show that _____

has more than ____?"

- "Tell me about what you did to figure out which has more."

**The educators support multilingual learners' engagement in comparing numbers by scaffolding English language development.**

- Ms. Greene provides an opportunity for all children to share their thinking by having the children share their ideas with a peer before opening the question up to the whole class. Offering Isla the option to count in Spanish or English, and then having all children count in both Spanish and English, Ms. Greene supports children in accessing mathematical knowledge in the home languages while also strengthening counting skills in English. Ms. Greene consults ELD preschool/TK learning foundations on sharing explanations and opinions (LLD:ELD.1.9.) and participating in conversations (LLD:ELD.1.10.) to support her teaching.
- Ms. King supported equitable mathematics talk by providing language frames (for example, "There are more ____ than ____."), having the children share with partners, and prompting them to listen carefully to their partners' reasoning. She prompts them to share their conceptual understanding with precision using specific mathematical language ("more than") and language for explaining their reasoning ("I know this because ____"). Ms. King addresses ELD standards on exchanging information and ideas (ELD.PI.1.1.), offering opinions (ELD.PI.1.3.), and listening actively (ELD.PI.1.5.) to support her teaching.

**Designated ELD (K–3 Educators):** In addition to integrated ELD (described above), during designated ELD time, Ms. King works with English learners to offer opinions (ELD.PI.1.3.), with a focus on understanding and using precise vocabulary (ELD.PI.1.12b.), including words to describe and compare quantity. Throughout the week, Ms. King uses playful activities using classroom objects, such as books and crayons, and items found in nature, such as leaves, rocks, or flowers, to support their use of language to compare numbers (for example, "bigger/smaller than," "more/less than") so the children are able to express themselves effectively and participate equitably in mathematical discussions.

# Key Area 2: Operations and Algebraic Thinking

**Content Connection: Taking Wholes Apart, Putting Parts Together**

In preschool and TK, children develop an understanding of foundational concepts of addition and subtraction, for example, understanding that adding increases the number of objects and taking away decreases the number of objects. Children in the preschool/TK years will build their understanding of addition and subtraction by working with small quantities and physical objects. During the early elementary years, children continue building their ability to add and subtract by working with larger numbers, multiple sets of numbers (for example, adding four two-digit numbers), and more complex tasks (for example, children may add and subtract within word problems containing extraneous information, requiring multiple steps, or using multiple operations to solve).

## Learning Progression Table 2.1: Represent and Solve Addition and Subtraction Problems

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.1. Demonstrate understanding that adding or taking away one or more objects from a group will increase or decrease the number of objects in the group.<br><br>PTK.Early.2.3. Solve addition and subtraction problems with a very small number of objects (sums up to 5 or 6), in the context of everyday situations. | PTK.Later.2.1. Demonstrate understanding that adding one or taking away one changes the number in a small group of objects by exactly one.<br><br>PTK.Later.2.3. Solve addition and subtraction problems with a larger number of objects (sums up to 10), in the context of everyday situations. | K.OA.1. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds, acting out situations, verbal explanations, expressions, or equations. | 1.OA.1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions. | 2.OA.1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions. | *Children in third grade continue to apply knowledge of addition and subtraction to solve various problems including problems involving time (3.MD.1.), finding the area of rectangles (3.MD.5-7.), and as a foundation for understanding multiplication and division (see standards 3.OA. 1–4.).*<br><br>*Children use these skills to solve word problems.* |

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## Learning Progression Table 2.1: Represent and Solve Addition and Subtraction Problems

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| | | | 1.OA.2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20.<br><br>1.OA.3. Apply properties of operations as strategies to add and subtract.<br><br>1.OA.4. Understand subtraction as an unknown-addend problem.<br><br>1.OA.5. Relate counting to addition and subtraction. | | 3.OA.8. Solve two-step word problems using the four operations (*addition*, *subtraction*, *multiplication*, and *division*). Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. |

## Learning Progression Table 2.2: Compose and Decompose Numbers

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.2. Demonstrate understanding that a set of objects is made up of smaller parts and that the whole set is bigger than its parts. . | PTK.Later.2.2. Decompose a set of objects into two small sets in more than one way (for example, decompose 5 into sets of 3 and 2, or 1 and 4). Combine two small sets to create a larger set (for example, 3 and 2 to make a set of 5). | K.OA.3. Decompose numbers less than or equal to 10 into pairs in more than one way.<br><br>K.OA.4. For any number from 1 to 9, find the number that makes 10 when added to the given number. | *In solving unknown whole numbers in addition and subtraction equations, children apply an understanding of how whole numbers can be broken into parts (for example, 11 can be decomposed into 8 and 3:* 1.OA.8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. For example, 8 + ? = 11). | *As children decompose numbers into hundreds, tens, and ones, they develop their understanding of place value.*<br><br>2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; for example, 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: | *Children in third grade move from a focus on whole numbers to also include fractions (for example, decomposing 1 into smaller parts:* 3.NF.1. Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by parts of size 1/b). |

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## Learning Progression Table 2.2: Compose and Decompose Numbers

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| 3 to 4 ½ Years Old | 4 to 5 ½ Years Old | Kindergarten | Grade 1 | Grade 2 | Grade 3 |
| | | | | a. 100 can be thought of as a bundle of ten tens—called a "hundred."<br>b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). | |

## Learning Progression Table 2.3: Add and Subtract Within a Range (5, 10, 20, 100, 1000)

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.3. Solve addition and subtraction problems with a very small number of objects (sums up to 5 or 6), in the context of everyday situations. | PTK.Later.2.3. Solve addition and subtraction problems with a larger number of objects (sums up to 10), in the context of everyday situations. | K.OA.2. Solve addition and subtraction word problems and add and subtract within 10.<br><br>K.OA.5. Fluently add and subtract within 5. | I.OA.6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction; and creating equivalent but easier or known sums. | 2.OA.2. Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.<br><br>2.NBT.5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. | 3.NBT.2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. |

## Learning Progression Table 2.4: Identify, Create, and Reason with Patterns

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.6. Notice and explore patterns in their environment and, with adult support, duplicate simple repeating patterns (for example, ABAB).<br><br>PTK.Early.2.7. Create, with adult support, a simple repeating pattern (for example, ABAB). | PTK.Later.2.6. Explore, extend, and duplicate a variety of repeating patterns (for example, AABBAABB, ABCABC) with adult support. Describe the repeating part of a pattern (pattern unit).<br><br>PTK.Later.2.7. Create a variety of repeating patterns (for example, AABBAABB, ABCABC) or re-create existing patterns using different objects. | *Patterns are foundational in various mathematical processes (for example, using patterns within the base-ten system when counting to 100:* K.CC.1. Count to 100 by ones and by tens). | *Patterns are foundational in various mathematical processes (for example, using patterns within the base-ten system to count and understand place value:* 1.NBT.1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral; 1.NBT.2. Understand that the two digits of a two-digit number represent amounts of tens and ones). | *Patterns are foundational in various mathematical processes (for example, using patterns within the base-ten system to count and understand place value:* 2.NBT.2. Count within 1000; skip-count by 2s, 5s, 10s, and 100s; 2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; for example, 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases). | 3.OA.9. Identify arithmetic patterns (including patterns in the addition table or multiplication table) and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends. |

# In-Practice Examples

## Learning Progression 2.1: Represent and Solve Addition and Subtraction Problems

### Shopping in the Classroom Market

The in-practice examples below demonstrate how educators support children in preschool and second grade to add and subtract in the context of their classrooms' pretend-play markets (refer to the learning progression Represent and Solve Addition and Subtraction Problems). While the in-practice examples below are grade specific, educators in other grades can adapt similar strategies to support children to represent and solve addition and subtraction problems.

### As you read,

Notice how children of different ages and in different grades:

- Develop increasingly complex number operation skills.
- Use various strategies to add numbers.

Notice how teachers:

- Connect mathematics to children's everyday lived experiences and cultures through meaningful mathematics investigations.
- Encourage all children to explore and express mathematics ideas through multiple representations.
- Facilitate mathematics discourse.
- Make modifications to the environment, materials, and/or instruction to ensure each and every child, including children with disabilities, can participate.

## How Many Tokens to Buy an Origami Crane and a Car? (Preschool)

Recently, Ms. Robles has added tokens and baskets to the dramatic play area in her preschool classroom (children ages three and four years old), creating a pretend market. She also added items that children in her classroom might like based on their interests, cultures, and lived experiences. For example, she has added an origami crane, stickers, toy cars, and items families have shared with the class including natural materials found in their community (such as rocks and shells). Each item has a tag with a written numeral and dots to visually represent the quantity of the written numeral—indicating how many tokens they need to get each item. One afternoon, Ms. Robles is playing alongside the children in their market. She notices that Nico has an origami crane and a toy car. Ms. Robles asks, "Do you have enough tokens for those?"

Nico responds, "Yep! I've got five tokens!"

"That is a lot of tokens. How many tokens in all do you need for both?" asks Ms. Robles.

Nico looks at the tags on the items and says, "This one's two tokens [pointing at the origami crane] and this one's one token [pointing at the toy car]."

"How many tokens is that all together?" Ms. Robles asks.

Nico takes his tokens and counts out two tokens for the origami crane and says, "This is two for the crane," then takes out one more token and says, "and this is one for the car, so that's one, two, three. I need three tokens. I have enough!"

By this time, Sarah has joined their conversation and is interested in what they are doing. Sarah holds up two fingers and says in Mandarin, "二 [two]." Adding another finger she says, "三 [three]!"

Modeling Sarah's gestures, Ms. Robles holds up two fingers and confirms, "That's right, Sarah! You need two tokens for the origami crane." Then adding another finger, she continues, "And you need one more for the car, which makes three tokens. Would you like to say that with Nico and me?"

Sarah says, "Yes!" Ms. Robles, Nico, and Sarah playfully chant, "You need one, two, three tokens for the crane and the car."

## How Much Does It Cost All Together? (Second Grade)

Recently, Mr. Jones has transformed the back area of his classroom into a pretend market. He has included several items, with price tags, that might be interesting or culturally meaningful to the children in his classroom for children to "buy" (for example, beads from the Caribbean, corn husk dolls, toy cars, and other items families have shared from home). Each child receives 30 pretend dollars. Mr. Jones also provides extra pretend money on the center table for children to access freely if they discover they need more during their play. During center rotations, a group of children are playing in the store. Akira wants to buy a toy dragon for $8, a toy car for $24, and a beaded necklace for $2. When she gets to the cashier, the children work together to find out how much the items will cost all together.

"I need to get more dollars. My stuff costs more than 30," Akira says. "I want to buy something for my brother, my sister, and this beaded necklace for my auntie—she has one just like it from Trinidad!"

Mr. Jones says, "I remembered your auntie showing her necklaces to the class when we were learning about patterns. I'm curious, Akira—how do you know you need more money?"

"'Cause eight and two makes 10. And then I add 10 to 24 and that makes 34. I only have $30," Akira responds.

**Educators can provide opportunities for children, including children with disabilities, to engage in mathematics learning in ways that meet their needs and preferences.** For example, educators can offer various modalities for children to represent and develop their understanding of mathematical concepts through symbols, verbal expression, visual representation, physical movement, or the use of assistive technologies.[14] Offering open-ended experiences—experiences that invite multiple approaches and solutions—is another way to support and include children's varying learning needs in mathematics learning. Open-ended experiences are flexible, allowing children to choose an approach that meets their skill level and ability. In addition, educators can provide specific supports to meet the needs of children with disabilities in accordance to their Individual Family Service Plan, Individualized Education Program, or 504 plan. For more information on supporting diverse learners, review the *Mathematics Framework*, Chapter 2: Teaching for Equity and Engagement.

Mr. Jones asks the cashier, “Akira only has $30. But her items total $34. How might we figure out how much more she needs?” As he says the word “more,” he gestures to the written word on the chart next to the cash register, which has terms he encourages the children to use at this station (for example, more, total, refund, exchange).

James, a child with a learning disability who uses a number line to help him add and subtract, is acting as cashier. He turns around and, using the number line taped to the wall, begins to count. He finds 30 on the number line and counts up until he gets to 34. “The difference is four. She needs four more dollars,” he says.

Akira uses her fingers to count from 30 to 34 and gets four more dollars from the center table.

## Highlights from the in-practice examples

**The educators support children to add and subtract in ways that foster feelings of belonging for all children.** They provide learning opportunities and materials that reflect and are responsive to children's daily lives, abilities, cultures, and languages by creating a pretend market in the classroom with items that are meaningful to children. The market experience is also something most children are familiar with when they visit various stores in the community with their families.

Here are some other examples of relevant and interesting experiences in their community that encourage children to represent and solve addition and subtraction problems:

- Children can set up other pretend markets (for example, a grocery store, lemonade stand, or restaurant) using ideas from businesses they've visited in their neighborhood.
- The class can go on a nature walk to observe different insects found outside. Later, children can use addition and subtraction to find out more about the insects living around their school (for example, "How many more insects did we find with wings than without wings?" or "How many insects did we find in all?").
- Children might use subtraction to determine how many more days of school there are left until the end of the school year.

**The educators encourage children to explore addition and subtraction using multiple representations.** In the preschool classroom, Ms. Robles includes dots beside numerals to provide multiple ways for children to understand quantity. The educators use open-ended questions, such as, "How might we figure out …" to encourage children to use an addition strategy of their choice—using a number line, their fingers, tokens, and play money, as well as grouping to make ten. By providing multiple ways for children to explore and express mathematics concepts, all learners, including children with disabilities, are able to engage in mathematics learning.

Here are some additional ideas for providing tools that encourage children to use multiple representations to solve addition and subtraction problems:

- Ten frames, base-ten blocks, counting chips, and play coins could be used for children to represent their addition and subtraction problem-solving.
- Children can use writing tools (for example, crayons/pencils and paper or a whiteboard and markers) to draw representations of addition and subtraction (including written equations, word problems, and drawings).

**The educators facilitate discourse about adding and subtracting.** They use prompts that encourage children to explain their thinking and talk through the strategies they

use to solve addition problems. For example, Mr. Jones asks, "How do you know …?" to encourage children to provide the rationale for their conclusions. Ms. Robles supports children's use of their home language to communicate mathematical concepts (for example, affirming Sarah's contribution in Mandarin by repeating her gestures and making the connection to the English mathematical vocabulary).

Here are some other ideas for how to support mathematics discourse related to addition and subtraction. Encourage children to express their ideas in ways that are most effective for them (for example, using their home language, gestures, or assistive technology).

- "Show me how you found the total."
- "How might we figure out what the difference is?"
- "What other way might you find out how many in all?"
- "What other places have you been to with your family where you need to figure out totals and differences?"

**The educators support multilingual learners' engagement in mathematics experiences by scaffolding English language development.**

- Ms. Robles acknowledges Sarah's home language and provides an opportunity for Sarah to develop English in a playful chant with her and Nico. Ms. Robles consults ELD preschool/TK learning foundations on understanding and using vocabulary (LLD:ELD.1.1., LLD:ELD.1.2.) and using complex sentence structures (LLD:ELD.1.4.) to support her interaction with Sarah.
- Mr. Jones provides a space for children to talk to each other and negotiate meaning using mathematical terms and posted a chart of specific words he encourages the children to use. Mr. Jones addresses ELD standards on using a wide variety of academic vocabulary to exchange information and ideas (ELD.PI.2.1.) and to communicate with precision (ELD.PI.2.12b.).

**Designated ELD (K–3 educators):** In addition to integrated ELD (described above), during designated ELD time, Mr. Jones works with English learners to use precise vocabulary (ELD.PI.2.12b.) and increasingly complex sentence structures (ELD.PI.2.5.–7.) when talking about mathematical concepts (for example, Mr. Jones makes up and teaches the children fun and playful chants, songs, and raps containing precise mathematical words and complex sentence structures).

# Key Area 3: Number and Operations in Base Ten


**Content Connection: Taking Wholes Apart, Putting Parts Together**

Children in preschool and TK are developing foundational skills related to place value in their explorations of composing and decomposing sets of objects. These explorations establish an understanding that a group of objects can be broken down into smaller parts or combined to make a larger set. In the early elementary years, children begin working with numbers greater than 10 and compose and decompose numbers using the base-ten system (for example, 28 is composed of 20 and 8). During this time, children's understanding of the base-ten system advances as they work with larger numbers and apply concepts related to place value to solve addition and subtraction problems.

## Learning Progression Table 3.1: Understand Place Value

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.2. Demonstrate understanding that a set of objects is made up of smaller parts and that the whole set is bigger than its parts.<br><br>PTK.Early.2.3. Solve addition and subtraction problems with a very small number of objects (sums up to 5 or 6), in the context of everyday situations. | PTK.Later.2.2. Decompose a set of objects into two small sets in more than one way (for example, decompose 5 into sets of 3 and 2, or 1 and 4). Combine two small sets to create a larger set (for example, 3 and 2 to make a set of 5). | K.NBT.1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, for example, by using objects or drawings, and record each composition or decomposition by a drawing or equation (for example, 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones. | 1.NBT.2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:<br><br>a. 10 can be thought of as a bundle of ten ones—called a "ten." | 2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; for example, 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:<br><br>a. 100 can be thought of as a bundle of ten tens—called a "hundred." | 3.NBT.1. Use place value understanding to round whole numbers to the nearest 10 or 100. |

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## Learning Progression Table 3.1: Understand Place Value

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| | PTK.Later.2.3. Solve addition and subtraction problems with a larger number of objects (sums up to 10), in the context of stories or everyday situations. | | b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.<br>c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones). | b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). | |

## Learning Progression Table 3.2: Use Place Value Understanding to Add and Subtract

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.2. Demonstrate understanding that a set of objects is made up of smaller parts and that the whole set is bigger than its parts.<br><br>PTK.Early.2.3. Solve addition and subtraction problems with a very small number of objects (sums up to 5 or 6), in the context of everyday situations. | PTK.Later.2.2. Decompose a set of objects into two small sets in more than one way (for example, decompose 5 into sets of 3 and 2, or 1 and 4). Combine two small sets to create a larger set (for example, 3 and 2 to make a set of 5).<br><br>PTK.Later.2.3. Solve addition and subtraction problems with a larger number of objects (sums up to 10), in the context of stories or everyday situations. | K.NBT.1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, for example, by using objects or drawings, and record each composition or decomposition by a drawing or equation (for example, 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones. | 1.NBT.4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, | 2.NBT.5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.<br><br>2.NBT.6. Add up to four two-digit numbers using strategies based on place value and properties of operations.<br><br>2.NBT.7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, | 3.NBT.2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.<br><br>3.NBT.3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (for example, 9 × 80, 5 × 60) using strategies based on place value and properties of operations. |

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## Learning Progression Table 3.2 Use Place Value Understanding to Add and Subtract

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| 3 to 4 ½ Years Old | 4 to 5 ½ Years Old | Kindergarten | Grade 1 | Grade 2 | Grade 3 |
| | | | ones and ones; and sometimes it is necessary to compose a ten.<br><br>1.NBT.5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used. | properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds. | |

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## Learning Progression Table 3.2: Use Place Value Understanding to Add and Subtract

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| 3 to 4 ½ Years Old | 4 to 5 ½ Years Old | Kindergarten | Grade 1 | Grade 2 | Grade 3 |
| | | | 1.NBT.6. Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. | 2.NBT.8. Mentally add 10 or 100 hundred to a given number 100–900. And mentally subtract 10 or 100 hundred from a given number 100–900.<br>2.NBT.9. Explain why addition and subtraction strategies work. Using place value and the properties of operations. | |

# In-Practice Examples

## Learning Progression 3.2: Use Place Value Understanding to Add or Subtract

### Planning for a Celebration: How Many Plates Are Needed?

The in-practice examples below demonstrate how educators support children in kindergarten and second grade to use place value to add or subtract (refer to the learning progression Use Place Value to Add and Subtract) in the context of a meaningful investigation—figuring out how many plates they will need for the celebration they are planning. While the in-practice examples below are grade specific, educators in other grades can adapt similar strategies to support children to use place value to add or subtract.

### As you read,

Notice how children of different ages and in different grades:

- Develop increasingly complex knowledge and skills related to place value, beginning with the foundational understanding in kindergarten that numbers can be decomposed into tens and ones.
- Use various strategies and manipulatives to add and subtract numbers using place value.

Notice how educators:

- Facilitate children's engagement in meaningful mathematics investigations and problem-solving.
- Encourage all children, including children with disabilities, to explore and express their ideas through multiple representations.
- Support mathematics discourse.

## How Many More Plates Do We Need? (Kindergarten)

Mr. Abara's kindergarten class is planning a celebration. "Have you ever been to a celebration or a gathering of friends and family?" Mr. Abara asks the class. After the children excitedly share their experiences, Mr. Abara asks, "What are some things we might need for our kindergarten celebration?" The children take turns suggesting items they might need, including food, balloons, plates, and music. Building off an example she shared earlier about a family gathering, Marisol says, "Mr. Abara, that time we had a party at my *tia's* [aunt, in Spanish] house, we used banana leaves as plates for our tamales!" Mr. Abara smiles and says, "Banana leaves do make great plates for tamales!" Mr. Abara acknowledges and validates the children's ideas. He explains that today they need to make sure they have enough plates so that each child receives one for their party. Mr. Abara sees this as a great opportunity for children to work on a real-world word problem. He writes the following on the board.

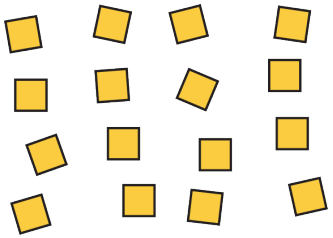
*We have 16 children.*

*We have 10 plates.*

Mr. Abara then poses the question to the class by saying, "We have 16 children in the class, but I only have 10 plates. *How many more plates do I need?*"

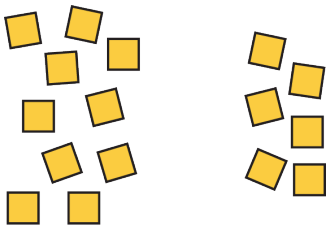
Mr. Abara invites the children to use the mathematics manipulatives in the baskets at their tables. "Just like we have practiced, you can use the base-ten blocks or the markers and whiteboards to help you figure this out."

Marisol decides on the unit blocks, grabs a handful of them from the basket, and begins to count until she reaches 16. "We need this many," she shares with Mr. Abara.



"Great! But I only bought a pack of 10. Since I don't have banana leaves to use like your *tía* [aunt] does, how many more plates do I need to buy? How about using the unit blocks to find out?" encourages Mr. Abara. Marisol starts counting 10 blocks, putting each one aside as she is counting.

After counting out 10 in Spanish and setting the extras aside, Marisol declares in Spanish and English, "I have a pile of 10 and a pile of *uno*, *dos*, *tres*, *cuatro*, *cinco*, *seis*—*six*!" (I have a pile of 10 and a pile of one, two, three, four, five, six—six!)



"So how many more plates do I need to buy?" prompts Mr. Abara.

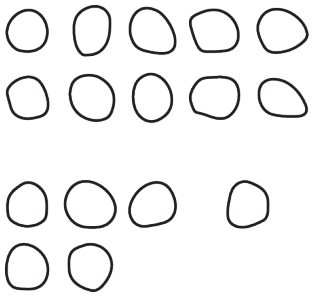
"We already got 10 plates," she explains, pointing to the pile of ten, "six more!" exclaims Marisol.

Mr. Abara smiles and gives Marisol a high five. He notices Nickolas is unsure of how to get started. Encouraging him, he says, "I know you love to draw, Nickolas, would you like to draw a picture to help figure this out? You could use the whiteboard and markers from the basket to try that."

"I do like to draw," Nickolas says and begins to draw plates on the whiteboard. "I drew ten plates. Now what?"

"We have ten plates. We need 16. How might you add to your drawing to show how many more plates we need?"

"I'll draw more plates until I get to 16." Counting as he draws, Nickolas says, "11, 12, 13, 14, 15, 16."



"That's right!" encourages Mr. Abara. "So, how many more plates do we need?"

"We need 16!" Nickolas shouts.

Mr. Abara smiles at his enthusiasm and says, "You used your drawing skills to solve the problem! We need 16 plates in total. Let's look at your drawing. You drew 10 plates on the top because we already have 10." Mr. Abara uses his hand to cover the drawing of the 10 plates and then asks, "How many more do we need so that we have a total of 16?"

Nickolas begins to count the plates on the bottom of the whiteboard, "One, two, three, four, five, six. We need six more plates," shares Nickolas. Mr. Abara offers a high-five and continues moving around the room.

When it appears that the children have had enough time to solve the problem on their own, he invites them to share their solutions in their table groups. He models how to use a language frame when sharing their solutions, "We need ____ plates. I know this because ____."

Mr. Abara concludes, "I listened to your conversations and noticed that you all decided that since we already have 10 plates, we need six more so that we have 16. Ten and six together make 16." He writes "16 = 10 + 6" on the board.

## Do We Have Enough Packs of Plates? (Second Grade)

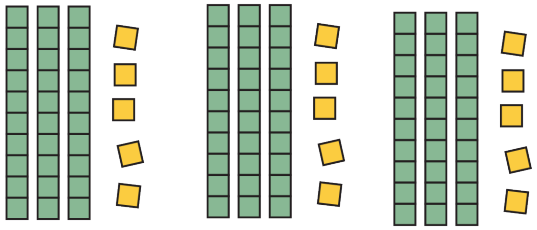
Ms. Jimenez's second-grade class is in charge of determining how many packs of plates are needed for the grade-level celebration. Yesterday, the class figured out how many children there are in total across all four classrooms. Now the children need to determine if Ms. Jimenez bought enough packs of plates. Ms. Jimenez sees it as a great opportunity for children to work on a real-world word problem. She posts the following word problem on the board and then reads it aloud.

*There are 104 children in total in all four of our second-grade classrooms added together. I bought three packs of plates. Each pack has 35 plates. How many plates do I have in total? Do we have enough plates so that everyone has a plate for the party?*

Ms. Jimenez lets the children know they will be working in groups to solve the problem. She invites the children to use the mathematics manipulatives in the baskets at their tables. The baskets include whiteboards and markers and packs of base-ten blocks. Ms. Jimenez notices some groups choose to use the base-ten blocks, while others prefer to work the problem out with the whiteboard and markers. She moves between groups to provide support.

"We want to see how many plates Ms. Jimenez bought, so 35 + 35 + 35," says Mustafa to Mira. "But sometimes when there are so many numbers, I mix them all up."

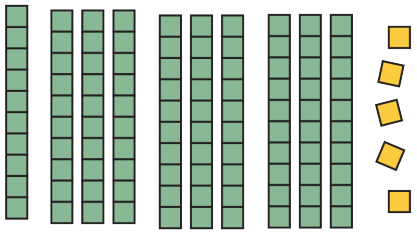
Mustafa, a child with a specific learning disability, chooses to use the base-ten blocks to help separate the tens and ones. He and his partner take turns creating groups of 35 using base-ten blocks, as follows:



"We have 10, 20, 30, 40, 50, 60, 70, 80, 90," Mustafa counts as he points to the ten rods, "I have 90 … and … 5, 10, 15." He pauses to think and then says, "90 plus 15." Mustafa pauses again as he considers how to add the numbers.

Ms. Jimenez says, "Think about the remaining 15 blocks. How can we use another ten rod?"

"There's another 10!" says Mira. "The 15 has a 10!" She takes out another ten rod and gathers five of the unit blocks as shown below:



"OK, let's count again … 10, 20, 30, 40, 50, 60, 70, 80, 90, 100," offers Mustafa.

"Plus, the five is 105!" says Mira.

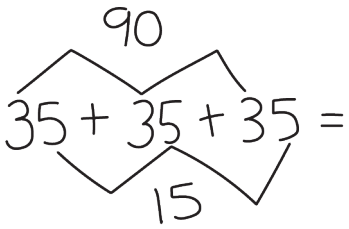
"So, we have 105 plates!" exclaims Mira.

While Mustafa and Mira are working through the problem using base-ten blocks, Ms. Jimenez notices Sam and Naomi chose a different way to solve the problem—adding the numbers using the whiteboard and grouping to get 105.

Sam begins by writing the problem on his whiteboard:

$$35 + 35 + 35 =$$

"Let's put our tens and ones together," suggests Naomi. "3 + 3 + 3 is 9, so 90 because of the tens place. 5 + 5 + 5 = 15."




Sam continues, "Now we need to add these—so let's add the tens—90 plus 1 is 91."

"No," says Naomi, "it's not a one—it's a ten. So, it's 90 plus 10 is 100."

The children determine that they have 105 plates, just enough for the party.

Ms. Jimenez brings the group back to the problem, "You all used different strategies to show that we have enough plates. How many extra plates will I have?"

"Just one!" exclaims Mustafa.

Next, Ms. Jimenez invites the children to design a poster to show how they solved the problem. "At the top of the poster, draw how you solved it, and at the bottom of the poster, write a short explanation of how you solved it." She tells them that after they are done, they will post their creations on the walls and do a gallery walk to see how others solved the problem.

## Highlights from the in-practice examples

**The educators support children to add and subtract using place value in the context of a meaningful, authentic investigation centered around a problem that matters to children**—making sure they have the right number of plates for their celebration. Not only are children interested in the problem, but they can also make personal connections to planning for a gathering or celebration—such as Marisol's connection to using banana leaves as plates at her tia's house.

Here are some other examples of authentic, problem-based investigations that encourage children to represent and solve addition and subtraction problems:

- While taking attendance, children figure out how many classmates are present.
- The children bring in recyclable items (for example, cereal boxes and plastic bottles) from home to use in a class project. The children use place value understanding to figure out the total number of recycled items they collected.
- After completing a book challenge, children are curious whether the class has read more fiction or nonfiction books. They can use their understanding of place value to add up the book titles in each category.

**The educators encourage children to explore and express their ideas related to adding and subtracting using place value through multiple representations such as base-ten blocks and drawings.** This approach allows children to use place value to solve problems in a way that matches their level of understanding. For children with disabilities, tactile manipulatives, like the base-ten blocks, further enhance engagement.

Here are some additional ideas to encourage different ways of representing and solving addition and subtraction problems:

- Consider using everyday materials and culturally unique items from home like counters, tokens, rocks, beads, marbles, dominoes, or pom-pom balls as concrete manipulatives to support counting and operations.
- Make available to children other forms of representation to approach a word problem, like ten frames or tally marks. Encourage children to choose the form of representation.
- Offer larger or textured base-ten blocks for easier manipulation of materials.
- Provide opportunities for children to explore place value using computer programs.

**The educators ask questions and probe children's ideas about using place value to add and subtract, engaging them in mathematics discourse.** For example, Ms. Jimenez asks, "How can we use another ten rod?" and Mr. Abara asks, "How might you add to your drawing to show how many more plates we need?"

Here are some other examples that probe children's thinking about using understanding of place value to add and subtract. Encourage children to share their ideas with each other and express their ideas in ways that are most effective for them (for example, using their home language, gestures, or assistive technology).

- "How else could you show me tens and ones?"
- "Check with your table partner to see if they solved this in a different way. Did you get the same answer?"
- "How else might you solve this problem?"
- "How do you know when you can exchange for a ten rod?"

**The educators support multilingual learners' engagement in mathematics experiences by scaffolding English language development.**

- Mr. Abara supports the children's abilities to engage meaningfully in conversations in English by providing a language frame for the children to discuss their strategies in their table groups. He consults ELD standards on exchanging information and ideas (ELD.PI.K.1.) to support his teaching.
- Ms. Jimenez supported children's abilities to explain their ideas, both orally in small groups and then in writing, and to interact with peers via written English in the preparation for and implementation of the gallery walk. Ms. Jimenez addresses ELD standards on exchanging information and ideas (ELD.PI.2.1.), interacting via written English (ELD.PI.2.2.), offering opinions (ELD.PI.2.3.), listening actively (ELD.PI.2.5.), and writing (ELD.PI.2.10.) to support her teaching.

**Designated ELD (K–3 educators):** In addition to integrated ELD (described above), during designated ELD time, the educators work with English learners to exchange ideas and opinions orally (ELD.PI.K.3., ELD.PI.2.3.). They focus on strengthening the children's abilities to engage in sustained dialogue in small-group conversations and use an expanded set of learned phrases (such as "I solved the problem by ____ because ____," "Your strategy helped me to ____," "I agree with you; however ____," and "Why did you ____?").

# Key Area 4: Measurement

**Content Connection: Discovering Shape and Space**

From a young age, children attend to various attributes of objects. In preschool, they describe and compare objects by different attributes such as length, weight, or capacity. During the early elementary years, children begin to use standard and nonstandard units of measurement to enhance the accuracy and precision of their descriptions and comparisons.

## Learning Progression Table 4.1: Describe, Compare, and Order Measurable Attributes

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.3.1. Demonstrate awareness that objects can be compared by length, weight, or capacity, by noticing differences in objects and communicating about their comparison.<br><br>PTK.Early.3.2. Order a few objects (for example, three) by length or other attributes (for example, height, capacity). | PTK.Later.3.1. Compare two objects by length, weight, or capacity (for example, putting objects side by side) and communicate about their comparison.<br><br>PTK.Later.3.2. Order a slightly larger number of objects (for example, four or five) by length or other attributes (for example, height, capacity). | K.MD.1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.<br><br>K.MD.2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter. | 1.MD.1. Order three objects by length; compare the lengths of two objects indirectly by using a third object. | 2.MD.4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit. | 3.MD.2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, for example, by using drawings (such as a beaker with a measurement scale) to represent the problem. |

## Learning Progression Table 4.2: Measure and Estimate Length

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.3.1. Demonstrate awareness that objects can be compared by length, weight, or capacity, by noticing differences in objects and communicating about their comparison. | PTK.Later.3.3. Measure length using concrete objects laid end-to-end, sometimes needing adult support.<br>*Note: Children may not yet understand that units need to be of equal length.* | K.MD.1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object | 1.MD.2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps. | 2.MD.1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.<br>2.MD.2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.<br>2.MD.3. Estimate lengths using units of inches, feet, centimeters, and meters. | *Children in third grade use measurement relating to length to solve problems involving area and perimeter (for example, 3.MD.7.a).*<br>3.MD.4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters. |

# In-Practice Examples

## Learning Progression 4.2: Measure and Estimate Length

### Does My Shadow Grow?

The in-practice examples below demonstrate how educators support children in later preschool/TK and second grade to measure length (refer to the learning progression Measure and Estimate Length) in the context of a meaningful investigation—exploring the length of their shadows. While the in-practice examples below are grade specific, educators in other grades can adapt similar strategies to support children to measure and estimate length.

### As you read,

Notice how children of different ages and in different grades:

- Develop increasingly complex measuring skills.
- Use various strategies and measuring tools to measure their shadow.

Notice how educators:

- Foster children's sense of belonging by offering meaningful, authentic investigations that connect with children's interests and cultures.
- Use observations of children's mathematics skills and knowledge to support and challenge their learning.
- Provide a mathematically rich learning environment by offering, and modifying, objects and tools for children to explore and make meaning of mathematical concepts.
- Facilitate mathematics discourse.

## How Big Is My Shadow? (Later Preschool/TK)*

Mr. Davis's TK class loves to play shadow tag outside. With help from their educator, the children observe that their shadows are different sizes. Mr. Davis sees this as a great opportunity to work on measurement with the class.

The class heads outside to measure their shadows with a partner. Mr. Davis provides bins of same-size objects such as wooden blocks and sticks, as well as measuring tapes and rulers for children to use.

"With your partner, you will take turns tracing each other's shadow. Everyone's shadow will look a little different—it is special and unique to you. One person will stand really still," Mr. Davis says as he models how to stand. "Then your partner will trace your shadow," he continues, as his classroom aide traces his shadow. "Then I want you to measure the length of your shadow. You can choose to measure with wooden blocks or sticks, like we did when we measured the height of the pumpkin the other day. You may also try to use a measuring tape or rulers," he concludes.

He has paired children in a way that supports their mathematical and language abilities. For example, he has partnered Diego and Maya together because they both use Spanish at home. Mr. Davis moves around, assisting groups as they trace and measure. He notices that Diego has put the blocks in a line next to Maya's shadow but with many large gaps between the blocks. Maya has placed her blocks back-to-back in a line, with no gaps.

Mr. Davis observes this and says, "Look at your shadow measurements. How do the blocks look different?" Knowing that Maya often prefers speaking in her home language, Spanish, Mr. Davis asks, "*Mira las medidas de las sombras. ¿En qué se ven diferentes los bloques?*" (Look at your shadow measurements. How do the blocks look different?)

"*Sus bloques están separados, y mis bloques están tocando.*" (His blocks are apart, and my blocks are touching) says Maya in Spanish, gesturing with her hands as she describes the blocks.

"When we measured how tall our pumpkins were last week, we stacked up the blocks, one on top of the other," says Mr. Davis, as he stacks one hand on top of the other to represent how they stacked the blocks. "How do you think the blocks should be when we measure our shadows?" asks Mr. Davis.

Moving his blocks together like Maya's, Diego says, "Like this!"

Noticing Diego's adjustment, Mr. Davis says, "That's right, Diego. You moved your blocks together. Now let's count to see how long each of your shadows is!"

Mr. Davis stops to listen in and support Maya to express herself

* This TK in-practice example can also apply to preschool programs serving four- and five-year-old children.

effectively. "This many," says Maya to her partner, Diego, as she points at the blocks that she has lined up beside the outline of his shadow. Mr. Davis encourages Maya to count the blocks and invites Diego to count along with her. Maya and Diego begin to count in Spanish, and Mr. Davis listens carefully, holding up a finger as Maya names each number. "*Diez!* [ten]," says Maya in Spanish when they finish counting.

Diego then counts the blocks he lined up to measure Maya's shadow and declares excitedly, "And Maya's shadow is 13 blocks long!"

## Choosing Tools to Measure Shadows (Second Grade)

Ms. Smith's second-grade class has been studying light and shadow. Children observe their shadows outside and notice that their shadows look different throughout the day. Ms. Smith invites an Elder from a local Tribal community to share about how people have used the sun's movement to tell time and understand seasons. After learning about how the sun moves throughout the day, the children wonder how the sun's movement might change their shadows. They are curious to measure the lengths of their shadows at different times of day.

The class heads outside in the morning. The children will work with a partner to trace and measure their shadows and then enter the measurement data in their math journals. Each group has chalk and a choice of measuring tape or a ruler. As one child holds still, their partner will trace their shadow and choose a measuring tool to measure how long their shadow is.

"Wow, Anna! The shadow of the wheels on your wheelchair looks so cool when you move!" says Titus as he and his partner, Anna, move to an open spot. Titus begins tracing Anna's shadow. "Your shadow is really long," says Titus, "It will take a lot of rulers to measure it." He starts measuring with the ruler, but he notices it doesn't work very well—the rulers are difficult to line up.

"Maybe we should use the measuring tape?" suggests Anna. "It's much longer than just a ruler," she explains.

"Yeah, I can try. Which numbers should I use on it?" asks Titus.

Ms. Smith, overhearing this conversation, says, "We have practiced reading inches on the measuring tape as well as feet with inches." She points to the inches and feet indicators as she talks with Titus. "Which one would you like to try?"

"I like to read just the inches," replies Titus. "So, your shadow from the bottom of your wheelchair to the top of your head is 75 inches long right now!" he says to Anna. Now it is Anna's turn to trace and measure Titus' shadow. Ms. Smith has provided her with a chalk holder that extends so she can easily draw on the sidewalk.

Later, the class heads outside to trace their shadows again with their partners.

“Oh no!” Titus exclaims. “Now you are only 44 inches long! Your shadow shrunk!”

Back in the classroom, Ms. Smith asks the children to do a quick write in their math journals about the data they gathered and the difference in length of the shadow they measured from the morning to the afternoon. Ms. Smith revisits information shared by the Elder and provides photos of sundials used in different cultures. Understanding more about how the sun’s placement in the sky affects the length of their shadows, the children are inspired to build their own sundial.

## Highlights from the in-practice examples

**The educators support children to measure in the context of meaningful, authentic investigations centered around a question that matters to children—**measuring their shadows. Children are also able to make connections to different cultures, for example, learning about how people have used the sun to tell time.

Here are some other examples of authentic problem-based investigations involving measurement that may be meaningful to children. Educators might strengthen cultural connections to learning experiences by using stories or inviting families and local experts to share information with the children.

- Children measure the growth of a plant over time.
- In preparation for a puppet show, the children measure to get the right length curtain.
- While creating Huichol yarn art,* children can measure the needed length of yarn.
- Children determine the area and perimeter of a garden bed in their school by first measuring its length and width.

**The educators use observations of children's mathematics skills and knowledge to support and challenge their learning.** For example, Mr. Davis notices differences in children's understanding of measurement. Diego left gaps between his blocks while Maya measured with no gaps between the blocks. He encourages the children to notice how their blocks look and think about how they have measured with blocks previously. In the second-grade classroom, Ms. Smith notices that Titus is having difficulty using the measuring tape. She reminds him that he can use inches or feet and shows him the different indicators on the tape measure. Then she encourages Titus to choose one of the units.

Here are some additional ideas related to observing how children measure and estimate length:

- As children work with measuring tools, notice how they use and communicate about the tool. Ask children to explain how they are using the tool. Their explanations might provide more information about their measurement knowledge and skills. As children use tools, notice how they read and communicate numerical information to describe the measurements.

* The Huichol are an Indigenous tribe of Mexico. Huichol yarn art, also known as yarn paintings, uses yarn to create images and designs on a wax-covered board.

- Allow children to choose which measuring tool to use and ask them to explain why they made that choice. Their choice of tool and explanations might provide more information on their understanding of measurement.

**The educators create mathematically rich environments by providing a variety of objects and measurement tools such as blocks, sticks, rulers, and tape measures.** By being given choices, children can measure the shadows in a way that makes sense to them based on their current understanding and further develop their measuring skills. The educator also provides an assistive tool, the chalk holder, to allow a child who uses a wheelchair to fully participate.

Here are some additional ideas related to the effective use of materials and the environment to support measurement.

- Invite families to share different ways they measure at home (for example, fingertip to shoulder is a yard, a palm full of spices is a tablespoon).
- Children can choose objects from their environment such as shoes, paper clips, string, or same-size rocks to measure with.
- Children might measure using objects with known lengths, for example, 3 x 5 index cards or 8 x 11 paper.

**The educators ask questions and probe children's ideas, engaging them in mathematics discourse.** For example, the educator asks, "How do you think the blocks should be when we measure our shadows?" and "How do the blocks look different?" The educator also supports mathematics discourse among peers by encouraging children that share the same home language to work together.

Here are some other examples to probe children's thinking and invite them to share their ideas with each other. Encourage children to express their ideas in a way that is most effective for them (for example, using their home language, gestures, or assistive technology).

- "What else can we use to measure?"
- "Why did you choose that tool to measure?"
- "How can you use that tool to measure?"

**The educators support multilingual learners' engagement in mathematics experiences by scaffolding English language development.**

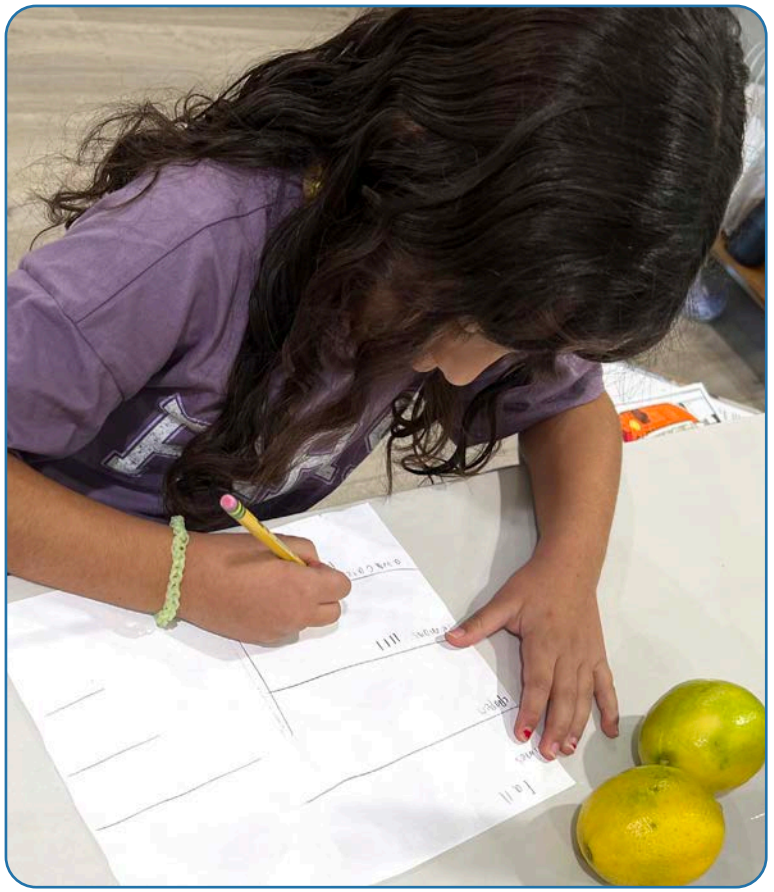
- Mr. Davis encourages Maya's sustained use of Spanish while also supporting her oral language development in English. He steps in at a "teachable moment" to help her more fully express her idea in a complete sentence, and he uses gestures to ensure Maya understands what she was saying, as well as to make the moment more engaging for her. Mr. Davis consults ELD preschool/TK learning foundations on understanding words (LLD:ELD.1.1.), using words (LLD:ELD.1.2.), and using grammatical features (LLD:ELD.1.3.) to support his teaching.
- Ms. Smith encourages the children to participate in collaborative conversations with peers by providing time for them to do a quick write of their findings before sharing orally with a partner. By asking the children to write in their math journals, Ms. Smith supports the children's writing development. Ms. Smith addresses ELD standards on exchanging information and ideas (ELD.PI.2.1.), offering opinions (ELD.PI.2.3.), listening actively (ELD.PI.2.5.), and writing (ELD.PI.2.10.) to support her teaching.

**Designated ELD (K–3 educators):** In addition to integrated ELD (described above), during designated ELD time, Ms. Smith works with English learners to contribute to class discussions with a particular focus on small-group conversations (ELD.PI.2.1.). Ms. Smith has an additional focus on supporting the children to engage in sustained dialogue with multiple exchanges by asking and answering questions to learn more about each other's ideas and building on each other's ideas using formulaic expressions, such as "Can you tell me more about ____," "I'd like to add on to what ____ said," and "I see what you mean, and I have another idea."

# Key Area 5: Data

**Content Connection: Reasoning with Data**

From a very young age, children notice similarities and differences in the attributes of objects. As they progress through toddlerhood and into preschool, they sort and classify objects based on different attributes (for example, color, size, shape). These actions are their early forms of categorizing and documenting data. Children in preschool will represent data using tally marks and pictographs and begin interpreting data to understand which groups have more or less. In the early elementary years, children build on these skills. They extend their representations of data to include different forms of data representation, such as picture graph, bar graph, or line plot, and extend their ability to use and interpret data to solve problems.



## Learning Progression Table 5.1: Classify, Represent, and Interpret Data

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.2.5. Notice similarities and differences in the attributes of objects. Sort and classify objects by one attribute in two or more groups. | PTK.Later.2.5. Sort and classify objects by one or more attributes, into two or more groups, with accuracy and flexibility. When sorting by two attributes, a child may first sort by one attribute and then by the second attribute. | K.MD.3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count. | 1.MD.4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another. | 2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph. | 3.MD.3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.<br><br>For example, draw a bar graph in which each square in the bar graph might represent 5 pets. |

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## Learning Progression Table 5.1: Classify, Represent, and Interpret Data

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.3.4. Use objects, tally marks, or pictographs to represent data in two groups, with adult support.<br><br>PTK.Early.3.5. Notice, with adult support, differences in the data of two groups and describe which one has more or less. | PTK.Later.3.4. Use objects, tally marks, or pictographs to represent data in two or more groups. Demonstrate understanding that each object, tally mark, or picture represents one data point.<br><br>PTK.Later.3.5. Describe and compare, with adult support, the number of data points in two or more groups. Determine which group has more or less. | | | | 3.MD.4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters. |

# In-Practice Examples

## Learning Progression 5.1: Classify, Represent, and Interpret Data

### Graphing Trees

The in-practice examples below demonstrate how educators support children in later preschool/TK and in third grade to classify, represent, and interpret data while they learn more about the trees that grow locally (refer to the learning progression Classify, Represent, and Interpret Data). While the in-practice examples below are grade specific, educators in other grades can adapt similar strategies to support children to classify, represent, and interpret data.


### As you read,

Notice how children of different ages and in different grades:

- Develop increasingly complex measuring skills.
- Use various strategies and measuring tools to measure their shadow.

Notice how educators:

- Create opportunities for children, including children with disabilities, to engage in meaningful mathematics investigations and problem-solving in ways that respond to individual learning needs.
- Encourage all children to explore and express their ideas through multiple representations.
- Engage children in mathematics discourse.
- Foster feelings of belonging for all children by providing learning opportunities and materials that reflect and are responsive to children's daily lives.

## Needles or Leaves? (Later Preschool/TK)*

Children in Ms. Scott's preschool class continue their study of trees by exploring the trees that grow in different areas of their school (for example, the schoolyard and garden area). When the children find a tree, they examine it, discuss its attributes (for example, the height, the type of the leaves, and the bark), and decide what kind of tree it is. Ms. Scott then takes a photo of the tree.

When they return to the classroom and discuss their observations, some children wonder if their school has more trees with needles than trees with leaves. Ms. Scott decides it is a good time to create a graph with the children. She prints out all the photos of the trees they found. The next morning, she invites the children to work together to sort the photos into two buckets. One bucket is labeled with the word "Needles" and an image of a conifer tree. The other bucket is labeled, "Leaves" with an image of a deciduous tree.

After gathering the class on the rug for circle time, Ms. Scott holds up the two buckets containing the sorted images of the trees. "Our trees are all sorted into ones that have needles—conifer trees, and ones that have leaves—deciduous trees. Today we are going to make a graph together. It is a great way to help us figure out whether we have more trees with needles or more trees with leaves growing around our school," says Ms. Scott.

**Educators can incorporate children's home, community, and cultural experiences and knowledge in mathematics learning.** To create inclusive, diverse, and culturally sustaining learning environments, educators can partner with families and communities to enhance mathematics learning. For example, educators can leverage families' funds of knowledge by inviting families to share materials, cultural stories, and experiences from home that can be incorporated into mathematics learning.[15] Incorporating children's cultures, interests, and prior experiences into their classroom supports their feelings of belonging and their development of positive identities as capable and competent mathematics learners.[16] Educators might include culturally relevant artifacts or objects (for example, counting candles on a menorah), local places (for example, using maps of local parks to support spatial thinking, or inviting families to explore their neighborhoods), and children's interests (for example, sports, hobbies, pets, local stores) into mathematical learning experiences.

Ms. Scott sets up a basic chart with the columns labeled "Needles" and "Leaves" to create a bar graph using photos of the trees. "You will pick an image out of the bucket and place it in the "Needles"

* This preschool in-practice example can also apply to TK programs serving four- and five-year-old children.

or "Leaves" column. Let's start with the conifer trees—the ones that have needles." She invites individual children to pick and place a tree image in the appropriate column.

After the children have placed the tree cards on the chart, Ms. Scott exclaims, "Wow! Look at this graph! Think to yourself, which type of tree do you think we have more of?" as she gestures with one hand down near the bottom of the graph and the other hand up near the top to show height. After providing a few moments for children to think, she says, "Turn and talk with your partner about which type of tree we have more of and tell them why you think that. You can start by saying, 'There are more ____ than ____.' Let's practice saying that together. So, if you think there are more trees with needles, you would say, 'there are more trees with needles than trees with leaves.' And then you would tell your partner why you think that."

Ms. Scott encourages children to work with partners that will support their language and mathematical abilities. For example, Jamal and Alon both speak Tagalog, so she has paired them together.

After the partners discuss, Ms. Scott invites the children to share what they talked about.

"There are more needles 'cause it's taller," says Jamal.

"*Mataás* [tall]," says Alon in Tagalog, his home language, moving his hand up.

Ms. Scott replicates Alon's gesture and says, "Tall," confirming his contribution to the discussion in English.

"You both said the 'Needles' column is taller. Jamal, what do Alon and you mean by taller?" asks Ms. Scott.

"The top one is taller," Jamal responds, as he points to the 'Needles' column.

"Ah, this column looks taller because we found more trees with needles than trees with leaves," says Ms. Scott.

"What about coconut trees?" Reyna asks, remembering the story of the coconut tree that her mother shared with the class last week.

"That's a great question, Reyna!" Ms. Scott replies.

"I have coconut trees by my house!" Jamal offers.

Ms. Scott begins to wonder with the children how palm trees are similar and different to the trees they found earlier that day. She invites the children to bring in leaves, needles, and palm fronds that they can find around their neighborhoods to further their study of trees.

## Creating Scaled Bar Graphs to Show Types of Trees Growing Around the School (Third Grade)

The children in Ms. Stacy's third-grade class have been studying trees native to California. The children start pointing out types of trees they see around their school and wonder which type of tree is most common. After discussing different ways they might figure out which type of tree is more common, they decide to explore the school and document the trees that they find. Before heading outside, Ms. Stacy invites the children to share some examples of trees they have seen around the school. Together, they decide to look for Arecaceae trees (palm trees), deciduous trees (trees with leaves), and conifer trees (trees with needles). They write down the three categories on a sheet of paper along with descriptions of the attributes that differentiate the types of trees.

Working in teams, the children explore around the school and use tally marks to indicate how many of each type they find. When the children get back to the classroom, they are excited to share the number of trees they found in each category.

To extend the investigation, Ms. Stacy plans for children to make graphs the next day to represent how many trees of each type they found. Ms. Stacy introduces the activity and a set of visual directions. "Yesterday, you all surveyed around the school to figure out which types of trees are growing. You came back with a lot of data. Today, we will create bar graphs. Graphs are a great way to organize data to help us interpret it so we can answer a question. What was the question we had about trees?"

"Which type of tree do we have more of?" suggests Ruby.

"That's right, Ruby. Let's see if you can work with your partner to create a graph that helps us answer this question. As you are working on your graphs, remember the parts of a graph and how we have set them up before."

Ms. Stacy references a vocabulary chart the children have been adding to over the past several days. The chart features words the children might use to discuss graphing along with illustrations and phrases next to each word to help the children remember what the words mean. She says, "I also want you to try to use mathematics language as you are talking with your partner." Ms. Stacy leads the class in a playful chant that contains the words "horizontal," "vertical," "bar graph," and other useful words, gesturing with her hands to communicate their meaning.

Ms. Stacy says, "You can look at the 'Types of Books' graph we created last week to help give you some ideas. I will be moving around the classroom to answer any questions you may have about setting up the bar graph."

As the children begin working, Ms. Stacy hears Nolan saying to Corinne, "We need numbers on one side and categories of trees on the other side. Do you know which goes where?"

"I think they can go in either place. Right, Ms. Stacy?" asks Corinne.

You are right, Corinne. They can go in either place. Where you put them will determine if you make a vertical or horizontal bar graph," shares Ms. Stacy, moving her hands up and down and side to side as she uses the words "horizontal" and "vertical." "Which kind of graph do you think you want to make?"

Nolan says, "I think we should make a horizontal graph. What do you think, Corinne?" Corinne agrees, and they continue working on their bar graph.

Ms. Stacy then moves on to Nathan, who is a child with autism. As the objective of the activity is not focused on collaboration today, Ms. Stacy allows him to choose to work with a team or individually, in alignment with his Individualized Education Program. Today he is working by himself and uses the visual directions and vocabulary chart for support when needed. Ms. Stacy hears Nathan say out loud, "Having to make this graph all the way up to 20 is going to take forever." He sees Ms. Stacy approach and asks, "Can I just label the numbers 6, 8, and 14, Ms. Stacy?"

"That would certainly be easier, but maybe not a very complete graph. Look over there at the graph we made for the types of books we have read. How did we number that?" prompts Ms. Stacy.

After thinking for a minute, Nathan says, "We skip counted by fives when we labeled it. We could skip count by twos for this one. That would help."

After complimenting his problem-solving, Ms. Stacy continues to move from group to group, assisting with setting up the vertical and horizontal axis, helping scale the number axis, and discussing how to display the bars on the bar graph.

When the groups start wrapping up, she brings the whole group back together. She notices that some groups created horizontal bars while some created vertical bars. "Nolan and Corinne and Nathan, can you hold up your graphs, please?" asks Ms. Stacy.

As they hold up their graphs, Ruby observes, "Those look totally different!"

"They do, Ruby! Take a look at Nolan and Corinne's graph and tell me how many palm trees they found."

"The bar goes to fourteen," responds Ruby.

"Isaiah, look at Nathan's graph. How many palm trees did they find?" asks Ms. Stacy.

"His says fourteen too," he responds.

"So, although these graphs look really different, how are they similar?" Ms. Stacy asks the class.

The children share their ideas and conclude that the direction of the bars, vertical or horizontal, doesn't change the information that the graphs reveal. "Looking at these graphs, can anyone tell me which trees are most common around our school?"

After volunteering, Ruby points at the bar for conifer trees. She starts at the top of the bar and slides her finger over to the scale on the left. “We have 17 conifer trees,” she shares.

“Great. We can use these graphs to answer so many different questions about the trees we studied!” concludes Ms. Stacy.

She invites teams to create and answer two additional questions related to the information on their bar graphs.

## Highlights from the in-practice examples

**The educators support children to classify, represent, and interpret data in the context of meaningful, authentic investigations centered around a question that matters to children:** "Which type of trees grow around our school?" Children are also able to make a connection to their lived experiences (for example, Reyna's reference to coconut trees). The educators support the inclusion of all children in representing and interpreting data, including those with disabilities and a range of learning needs, by adhering to UDL principles and providing accommodations when needed (for example, offering Nathan, who has autism, the option to work individually and use visual directions and the vocabulary chart).

Here are some other examples of authentic problem-based investigations that might encourage children to classify, represent, or interpret data:

- Children can collect and display data to represent the different ways classmates get to school (for example, classmates walk to school, take a car to school, or ride a bus to school).
- After collecting flowers at school, children can display data about various attributes of the flowers.

**The educators encourage children to explore different ways to classify, represent, and interpret their data** such as photos of trees, vertical and horizontal bar graphs, and different scaling methods.

Here are some examples of ways to encourage children to explore and express their understanding of classification, data representation, and data interpretation:

- In addition to creating bar graphs, children can display this data using tally marks, pictographs, or pie charts.
- Children can explain their interpretation of the chart with mathematical vocabulary that makes sense to them such as vertical, horizontal, up, down, across, taller, and shorter.
- Children can design graphs on a computer. This might provide opportunities for children with different kinds of abilities to be successful.

**The educators ask questions and probe children's ideas about classifying, interpreting, and representing data, engaging them in mathematics discourse.** For example, the educator asks, "Jamal, what do Alon and you mean by taller?" and "… how are they similar?" The educator also supports mathematics discourse by partnering children that share a home language—allowing them the opportunity to communicate in the language that is most effective for them (for example, Jamal and Alon both speak Tagalog, so they discuss the graph together in the language of their choice).

Here are some other examples that probe children's thinking about classification, data representation, and data interpretation. Encourage children to share their ideas with each other, expressing themselves in ways that are most effective for them (for example, using their home language, gestures, or assistive technology).

- "What can you learn from this graph?"
- "Which category has the highest bar?"
- "Why do you prefer the vertical bar graph?"
- "How else could you display this data?"
- "What questions might we be able to answer using this data?"

**The educators supported multilingual learners' engagement in mathematics experiences by scaffolding English language development.**

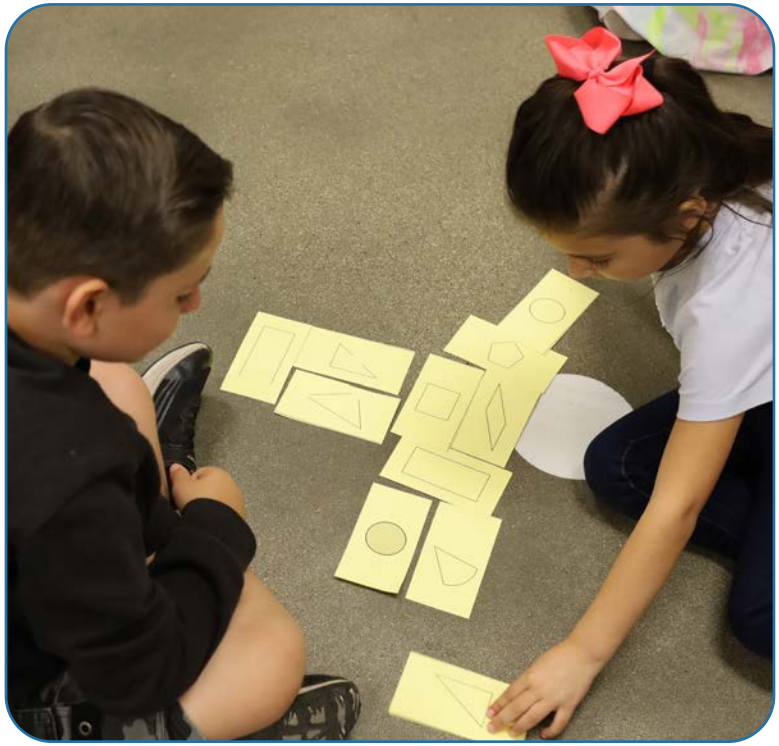
- By providing a language frame with a complex sentence and having the children turn and talk to a partner before asking the whole class to share their ideas, Ms. Scott provides an opportunity for all children to engage in meaningful talk in which they develop the oral language needed to participate more fully in the whole-group discussion. Ms. Scott consults ELD preschool/TK learning foundations on participating in conversations (LLD:ELD.1.10.) and using complex sentence structures (LLD:ELD.1.4.).
- By providing a chart with the mathematics terms that included illustrations and phrases, having the children use the words in a playful way, and then having the children use the terms meaningfully in their partner and whole-group discussions, Ms. Stacy supported the children's mathematics vocabulary development. Ms. Stacy consulted ELD standards on using domain-specific vocabulary (ELD.PI.3.12.).

**Designated ELD (K–3 educators):** In addition to integrated ELD (described above), during designated ELD time, Ms. Stacy works with English learners to contribute to peer and class discussions (ELD.PI.3.1.). They focus on strengthening the children's use of precise mathematics vocabulary (ELD.PI.3.12.) and abilities to connect multiple ideas using complex sentences (ELD.PII.3.6.). They teach the vocabulary explicitly and engage the children in word play games and playful chants to practice the mathematics vocabulary. Ms. Stacy also focuses on supporting the children to enrich their sentences by expanding noun phrases in a variety of ways (ELD.PII.3.4.), such as by adding comparative and superlative adjectives to nouns. The children first expand noun phrases orally through interactive games and then transfer this learning to writing.

# Key Area 6: Geometry and Spatial Thinking

**Content Connection: Discovering Shape and Space**

From a young age, children begin to explore shapes and space. In preschool, children learn the names of various shapes and shape attributes. Children also learn how to manipulate shapes, including the effects of composing and decomposing them. In the early elementary years, children's understandings of shapes advance to include a larger variety of two- and three-dimensional shapes and more complex ways to reason with shapes (for example, partitioning).



## Learning Progression Table 6.1: Identify, Describe, and Reason with Shapes and Their Attributes

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.4.1. Identify familiar two-dimensional shapes, such as a circle, square, triangle, and rectangle. Note: May not identify a nontypical version of a shape (for example, a square turned so that the point is down—a diamond). | PTK.Later.4.1. Identify, describe, and construct different shapes including variations of a circle, square, triangle, rectangle, and other shapes. Use informal language to describe defining properties of a shape (for example, sides, corners, round). | K.G.1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to. | 1.G.1. Distinguish between defining attributes (for example, triangles are closed and three-sided) versus non-defining attributes (for example, color, orientation, overall size); build and draw shapes to possess defining attributes. | 2.G.1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes. | 3.G.1. Understand that shapes in different categories (for example, rhombuses, rectangles, and others) may share attributes (for example, having four sides), and that the shared attributes can define a larger category (for example, quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of |

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## Learning Progression Table 6.1: Identify, Describe, and Reason with Shapes and Their Attributes

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.4.2. Occasionally identify a few familiar three-dimensional shapes using informal names (for example, saying "ball" when referring to a sphere).<br><br>PTK.Early.4.3. Compare two-dimensional shapes of different sizes and orientations to determine whether they are the same shape. | PTK.Later.4.2. Identify a few familiar three-dimensional shapes, such as spheres, cubes, and cylinders. Note: Sometimes still use informal names (for example, ball, square box, tube).<br><br>PTK.Later.4.3. Compare two-dimensional shapes of different sizes and orientations to determine whether they are the same shape. Identify similarities and differences in the | K.G.2. Correctly name shapes regardless of their orientations or overall size.<br><br>K.G.3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid"). | | | quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories. |

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## Learning Progression Table 6.1: Identify, Describe, and Reason with Shapes and Their Attributes

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.4.5. Identify some positions of objects and people in space such as in/on, under/over, up/down, and inside/outside.<br><br>PTK.Early.4.6. Rely on trial and error to determine how objects move in space or fit in different locations (for example, try to fit an object into a hole by rotating, flipping, or sliding the piece in different orientations until it fits). | properties (number of sides or vertices) of two different shapes.<br><br>PTK.Later.4.5. Identify positions of objects and people in space, including in/on, under/over, up/down, inside/outside, near/far, next to, beside/between, and in front/behind.<br><br>PTK.Later.4.6. Rotate, flip, or slide objects to solve a problem without relying as much on physical trial and error (for example, rotate an object before fitting it into a hole). | K.G.4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (for example, number of sides and vertices/"corners") and other attributes (for example, having sides of equal length). | | | |

## Learning Progression Table 6.2: Create, Compose, and Partition Shapes

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| **3 to 4 ½ Years Old** | **4 to 5 ½ Years Old** | **Kindergarten** | **Grade 1** | **Grade 2** | **Grade 3** |
| PTK.Early.4.4. Use two- or three-dimensional shapes to represent different elements of a picture or design (for example, adding a circle in a corner to represent the sun). | PTK.Later.4.4. Combine different two- or three-dimensional shapes to create a picture or design (for example, make a house with two blocks shaped like rectangular prisms and one shaped like a triangular prism). | K.G.5. Model shapes in the world by building shapes from components (for example, sticks and clay balls) and drawing shapes.<br><br>K.G.6. Compose simple shapes to form larger shapes. For example, "Can you join these two triangles with full sides touching to make a rectangle?". | 1.G.2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. | 2.G.2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.<br><br>2.G.3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape. | 3.G.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape. |

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## Learning Progression Table 6.2: Create, Compose, and Partition Shapes

| Preschool/Transitional Kindergarten Learning Foundations | | Common Core State Standards | | | |
|---|---|---|---|---|---|
| 3 to 4 ½ Years Old | 4 to 5 ½ Years Old | Kindergarten | Grade 1 | Grade 2 | Grade 3 |
| | | | 1.G.3. Partition circles and rectangles into two and four equal shares, describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of. Describe the whole as two of or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares. | | |

# In-Practice Examples

## Learning Progression 6.2: Create, Compose, and Partition Shapes

### Construction Zone!

The in-practice examples below demonstrate how educators support children to create, compose, and partition shapes as they construct different types of houses in later preschool/TK and first grade (refer to the learning progression Create, Compose, and Partition Shapes). While the in-practice examples below are grade specific, educators in other grades can adapt similar strategies to support children to create, compose, and partition shapes.

### As you read,

Notice how children of different ages and in different grades:

- Develop an increasingly complex understanding of shapes.
- Use various strategies and manipulatives to create, compose, and partition shapes.

Notice how educators:

- Use observations of children's mathematics skills and knowledge to support and challenge their learning.
- Foster feelings of belonging for all children by providing learning opportunities and materials that are responsive to their daily lives, abilities, cultures, and languages.
- Ask questions and probe children's ideas, engaging them in mathematics discourse.

## Tangram Houses (Later Preschool/TK)*

After observing the children's interest in the building construction around the school, the class reads together a book about homes across the world. Inspired by the book, children have been busy building houses with shapes. After working on building tangram pictures this week with the specific shapes outlined, Ms. Garcia concludes that the children are ready for a challenge. She reviews the names of the shapes, using a poster to provide a visual for children to refer to. Knowing that children in her classroom speak English and Spanish, she also reviews the word for "house" in both languages. Then she gives each child a picture of a house without the shapes outlined within, along with a set of shapes. This is a great opportunity for the children to observe that different shapes can be used to create the house and use language to communicate mathematical ideas and explain their thinking. As Ms. Garcia moves around the room to observe and support the children, she intentionally looks for opportunities to support the children to use precise mathematics terms and expand on their explanations about how they made the houses. The image below shows the silhouette of the house Ms. Garcia has provided as a challenge for children.

"I can do this!" Leah confirms as she gets to work. "This one doesn't fit," she says.

"Which one doesn't fit?" asks Ms. Garcia.

"This orange one," Leah says.

"You're having trouble getting the orange parallelogram to fit," Ms. Garcia says.

"Yeah! The parallelogram," Leah says.

"What would happen if you tried to rotate, or turn, the parallelogram?" Ms. Garcia asks.

After trying multiple times, Leah exclaims, "Rotate … rotate ... I got it! Look!"

* This preschool in-practice example can also apply to TK programs serving four- and five-year-old children.

Below is an image showing the shapes Leah uses to construct her house.

"This is harder than before!" exclaims Simon. "I can't remember what shapes to use."

Ms. Garcia suggests, "There are lots of ways you can build your house. Why don't you start from one place and build around it? Which shapes do you want to use first?"

Simon says, "I want to use triangles," and focuses first on the bottom part of the house. Gaining confidence, he moves on to the roof and chimney parts.

Below is an image showing the shapes Simon uses to construct his house.

Ms. Garcia, noticing they are both done, asks, "What is different about how you built your houses?"

"I used lots of different shapes, like triangles and a square, and the para … parallelogram. He only used triangles!" shares Leah.

"Yeah!" adds Simon. "Look at your chimney—it's a square and mine is two triangles."

"Great observations! Leah used squares, triangles, and a parallelogram. Simon used all different-sized triangles," says Ms. Garcia. Ms. Garcia notices that Carmen has stopped building and says, "What shape are you looking for, Carmen?"

"*No hay cuadrados*," (There are no squares) says Carmen in Spanish.

"Oh, interesting! No squares," responds Ms. Garcia.

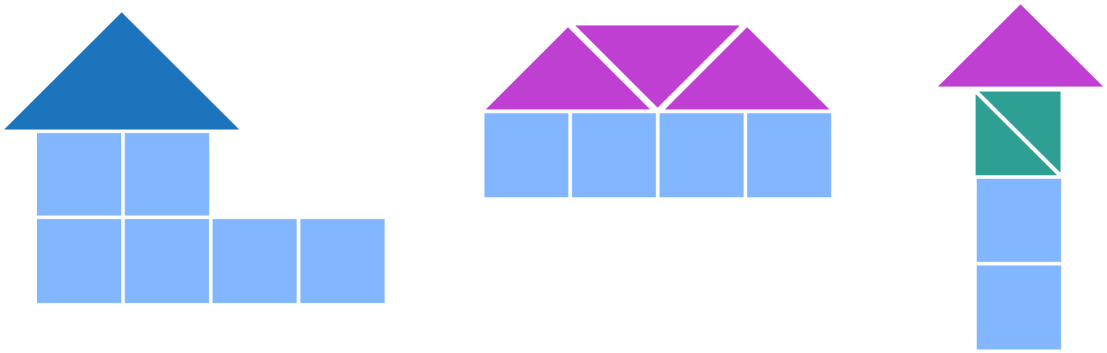
"No squares," says Carmen.

"What other shapes can you use instead?" asks Ms. Garcia.

After trying to fit different shapes in the square spot, Carmen responds in Spanish, "*Triángulos* [triangles]!"

"Great! Guess what? '*Triángulos*' in Spanish is 'triangles' in English. They sound almost the same!" says Ms. Garcia.

Children who have finished creating their houses early are invited to create their own designs. Ms. Garcia encourages the children to share about the different houses they have seen in their neighborhood and think about the different houses they observed in the book they shared earlier. After talking about the varied shapes, sizes, and colors of houses, Ms. Garcia invites them to create their own houses. She encourages them to talk about the shapes they use to make the different parts of the house. The image below shows some different ways children make houses.



## How Do We Build Without Rectangular Prisms? (First Grade)

After observing the children's interest in the building construction around the school, the class reads together a book about homes across the world. Inspired by the book, children have been busy building houses. Ms. Coleman further extends their interest in houses by sharing pictures of the many different houses throughout the community. She wants to give the children an opportunity to build one they are interested in. Ms. Coleman provides three-dimensional wooden shapes for building houses during center rotations. It offers a great opportunity to continue working on three-dimensional shape names as well as creating

new shapes from composite shapes. Before starting the activity, Ms. Coleman reviews and posts a chart with the shape names and drawings of the shapes.

As each child gets to work on their house, Ms. Coleman moves around the room.

Samira has been working hard on building an adobe home. To encourage using the names for the three-dimensional shapes, Ms. Coleman asks, "What shapes are you using to build your adobe home, Samira?"

"Most of them are the square and rectangle blocks," she responds as she finishes up her walls.

"Are you talking about these shapes that have square and rectangular faces?" Ms. Coleman asks. "Take a look at our shape name chart—what are the names of these shapes?" Ms. Coleman asks, pointing to the chart.

Samira reviews the chart and excitedly says, "The rectangular prisms!"

"Yeah, but now there aren't any of those prisms left for me! How am I supposed to build my apartment building?" asks Aaron.

To encourage this important question raised by the children, Ms. Coleman asks, "Let's try to find another way to make the square and rectangular prisms you were using with the shapes you do have. Let's experiment!"

While the children continue to create their buildings and problem solve, Ms. Coleman notices Shanti combining shapes to make other shapes. "Shanti, how did you make your walls without the rectangular prisms?"

Shanti, who has a speech impairment, reaches for two triangular prisms, holds them up, and shows Ms. Coleman how she can put them together to create a rectangular prism.

"Great thinking, Shanti! You used two triangular prisms to create a rectangular prism!" notes Ms. Coleman.

"Cool!" says Aaron, "Can you show me how you did that?"

Shanti shows Aaron how she used two triangular prisms to create a rectangular prism. "Shanti," Ms. Coleman asks, "is the rectangular prism you created the same as the rectangular prisms that Samira used for her adobe-style house?" Shanti thinks for a while, and then, borrowing a rectangular prism from Samira, she compares the two rectangular prisms. The children observe the two shapes and start noticing how they are similar and why the two triangular prisms can work to create a rectangular prism.

## Highlights from the in-practice examples

**The educators use observations of children's mathematics skills and knowledge to support and challenge their learning.** For example, in the preschool classroom, Ms. Garcia notices that children have mastered building with tangrams when the shape outlines are provided, so she offers them a challenge by providing only the silhouette. She extends this experience further by inviting children to create their own house designs. In the first-grade classroom, Ms. Coleman observes Aaron's need for support. Noticing Shanti's understanding of how to compose three-dimensional shapes, Ms. Coleman encourages her to explain, through gestures, her work as peer support for Aaron.

Here are some additional ideas related to observing how children create, compose, and partition shapes:

- As children work with shapes, notice what they are doing as well as what they are saying.
- In addition to intentional mathematical lessons and experiences, observe children throughout the day to identify moments when children authentically create, compose, or partition shapes.

**The educators foster feelings of belonging for all children by providing learning opportunities and materials that reflect and are responsive to children's daily lives, abilities, cultures, and languages.** The educators use a book to introduce various types of homes that children may live in across a variety of cultures and pictures of different houses in their community. Children are also invited to build houses of their choice. Some children choose to build houses they observed in the book, and others might build houses they have observed from their lived experiences (for example, some children build an adobe, and others build an apartment).* The educators are also responsive to children's various abilities (for example, Shanti, a child with a speech impairment, benefits from the tactile nature of the experience and the ability to use physical objects and gestures to communicate). Ms. Coleman notices Shanti's understanding and invites her to share with others.

Here are some other ideas for how to support feelings of belonging for all children related to creating, composing, or partitioning shapes:

- Engage children in designing and creating. For example, children might design garden beds for the school, create

* Educators should be sensitive to different children's experiences related to homes and housing. If there are children experiencing homelessness or housing insecurity, this may not be an appropriate experience.

origami art, or use recycled materials to create an instrument. Allow time for each child to share about the structures or designs that they create. Encourage them to notice and discuss the different ways they might have composed or partitioned shapes as part of their design.

- Invite families to share building materials and everyday objects such as paper towel rolls, empty water bottles, and empty cereal boxes for children to use to create, compose, or partition shapes.
- Provide opportunities to build tangrams in different ways to support diverse abilities, for example, using a computer or, for children with visual impairments, magnetic tangrams.

**The educators ask questions and probe children's ideas, engaging them in mathematics discourse—in English and the home language.** For example, Ms. Coleman says, "Let's try to find another way to make the square …" and, "… how did you make those walls?" Educators also support the use of home languages, allowing children to express their understanding of mathematical concepts in a way that is most effective for them (for example, when Carmen uses Spanish to discuss shapes as she builds her tangram).

Here are some other examples that probe children's thinking about creating, composing, or partitioning shapes. Encourage children to share their ideas with each other in ways that are most effective for them (for example, using their home language, gestures, or assistive technology):

- "What shapes can you use to create a rectangle prism?"
- "If you cut this square in the middle, what shape will each half be?"
- "Tell me about the shapes that you used."

**The educators support multilingual learners' engagement in mathematics experiences by scaffolding English language development.**

- Ms. Garcia supports the children's vocabulary development by making her expectations for using vocabulary clear before- starting the activity (for example, posting a chart with shape images and names) and prompting the children to use the vocabulary during the activity. Ms. Garcia affirms Carmen's use of Spanish and points out a cognate pair (triángulos/triangles), which supports Carmen's multilingual development. Demonstrating for children how to use new mathematics terms in English also supports Carmen's vocabulary knowledge. Ms. Garcia consults ELD foundations

on understanding and using words (LLD:ELD.1.1., LLD:ELD.1.2.) to support her teaching.

- Ms. Coleman supports children's vocabulary development by posting a chart of mathematics terms, making her expectation for using the vocabulary explicit, and supporting children to use the terms while building.

Ms. Coleman consults ELD standards on selecting domain-specific vocabulary (ELD.PI.1.12b.) to support her teaching.

**Designated ELD (K–3 educators):** In addition to integrated ELD (described above), during designated ELD time, Ms. Coleman works with English learners to develop proficiency in using mathematics vocabulary (ELD.PI.1.12b.). Ms. Coleman teaches the words explicitly and then engages the children in word play by, for example, singing songs with the words and playing games in which the children would have to use the words.

# Appendix A

## Supporting English Language Development Across the P–3 Continuum

Multilingual children in California schools and early education programs are developing proficiency in both English and one or more other languages. Supporting multilingual learners' language development involves promoting the continued development of the home language and English. Regardless of the language or languages used during instruction, linguistically sustaining educators are aware that multilingual children's home languages are valuable personal, intellectual, and community resources.* They recognize that children are at risk of losing their home language competence as they develop their English language skills. Linguistically sustaining educators also acknowledge the importance of supporting the development of the heritage language for children from Native nations and tribal communities that are engaged in language revitalization efforts. Children's home language development is foundational to learning additional languages. Educators can create classroom environments that invite children's use of home languages and heritage languages throughout the day. Educators can also encourage families to maintain and continue to develop the home languages and heritage languages in their communities as children add English to their linguistic repertoires. Overall, children's multilingualism should be valued as an asset.

For many multilingual learners, the P–3 school environment is the primary context for learning English. English language development (ELD) instruction provides equitable access to, and meaningful participation in, learning activities conducted in English and supports children's steady progress toward full proficiency in the English language. In preschool and transitional kindergarten (TK) programs, educators integrate ELD throughout daily routines and classroom activities. In K–12, educators take a comprehensive approach to ELD instruction, which includes both integrated ELD (instruction that occurs throughout the day in all content areas) and designated ELD (dedicated ELD instructional time).**

### Teaching Strategies to Support English Language Development

Educators use engaging and interactive teaching strategies to support ELD using a culturally and linguistically sustaining approach. While particularly helpful when supporting English learners, the following strategies can benefit all children's learning:

* Chapter 3 in *Improving Education for Multilingual and English Learner Students* provides an overview of language acquisition program models and pedagogical approaches.

** Integrated and designated ELD are explained in chapter 2 in the *English Language Arts/English Language Development Framework* (CA ELA/ELD Framework). Content referencing transitional kindergarten should be used alongside the *Preschool/Transitional Kindergarten Learning Foundations*.

- **Hands-on learning:** Engage children in experiential learning activities where they interact with the content actively and in collaboration with peers. For example, children engage in a science investigation in which they use three-dimensional models to share what they have observed or use art to create a visual representation of what they learned about a topic. These opportunities deepen understanding of content and provide authentic opportunities for children to practice their English.
- **Props and concrete objects:** Use concrete objects and materials, including props, visual representations, and costumes, to help children learn new vocabulary in English and make meaning about new topics. For example, invite children to reenact a story or invent a new story using props. Using these materials helps children deepen their understanding of content presented in English. It also presents an opportunity to practice using content-related language they are learning in English, including vivid vocabulary and dialogue.
- **Protocols and discussion norms:** Provide protocols with clear roles and steps or norms for participating in discussions in English. Providing protocols or discussion norms helps children engage meaningfully in productive discussions, reinforces their discussion skills such as turn taking, and strengthens their English language proficiency as speakers and listeners.
- **Language frames:** Provide and model the use of language frames in English (for example, "I think ____, because ____."). Language frames should be developmentally and linguistically appropriate and over time can contain increasingly complex grammatical structures or reference word banks for specific parts of speech (for example, "I think the character was behaving ____ + adverb because ____." *Adverb word bank: selflessly, recklessly, selfishly, angrily*). Language frames introduce language patterns and academic terms that children can then use, unprompted, in spoken and written English. This helps them to express themselves effectively and expands their grammatical knowledge.
- **Think-pair-share:** Provide children opportunities to discuss an idea or solution to a problem with a partner in English after they have had a chance to think about it independently. This peer-to-peer discussion strengthens children's use of English and expands their conceptual understanding. Think-pair-share also supports children in rehearsing the language they might later use in a whole-group setting. It is important to structure the think-pair-share with a protocol that sets expectations for listening, speaking, and using conversation norms.
- **Vocabulary instruction:** Use explicit instructional routines to help children learn new academic vocabulary in English and model appropriate use of vocabulary. Encourage them to use the words over time in various activities, such as in the context of hands-on learning activities or free play with peers. This instruction helps children learn the words deeply so they can use them intentionally in speaking and writing.
- **Songs, chants, and gestures:** Use songs or chants in English about a novel concept or topic, using new vocabulary and gestures. This use reinforces children's conceptual understandings and strengthens their ability to use new vocabulary in English.

- **Wide reading:** Provide high-quality children's literature from different genres and informational texts on diverse topics to help build children's knowledge about language and content in English. To ensure children see themselves reflected and represented in texts, choose texts that are relevant to children's families and communities and are written by authors from those communities. Support children through educator read-alouds, independent reading, and shared reading experiences with other children (for example, engaging in small-group discussions about a text). These practices enhance children's literacy skills, vocabulary expansion, and content knowledge.
- **Graphic organizers and visual supports:** Incorporate graphic organizers with language supports and visuals as scaffolds during learning activities conducted in English. Educators might provide graphic organizers that contain visuals or descriptions of key vocabulary in a text or language frames to support children's discussions or written responses to the text. These materials support children's comprehension of English texts and productive spoken and written language in English.
- **Translanguaging:** Offer opportunities for children to leverage all their linguistic resources, including the home language or other languages the child uses, whether instruction is provided in English, their home language, or another language the child uses. Translanguaging, when children combine and integrate languages they know when communicating, is a natural part of being multilingual that helps with learning. For example, educators can pair children who share the same home language in a think-pair-share activity to discuss an idea before sharing it in a small group.

# Appendix B

## Additional Resources

**Key Area 1: Counting and Cardinality**

For extended vignettes on how children compare numbers and use place value, refer to the following:

*California Preschool Curriculum Framework, Volume 1* (Additional resources from the *Preschool Curriculum Frameworks* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "Mathematical Reasoning in Action: Who Has More Cars?" page 243
- "Mathematical Reasoning in Action: How Many Boys? How Many Girls?" page 253

*Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (Transitional Kindergarten vignettes and snapshots from the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "How Do Students in Transitional Kindergarten Through Grade Two Learn to Compare and Order Numbers?" Chapter 3
- "The Pocket Game" Chapter 6
- "Number Talk with Addition, Grade Two" Appendix C

**Key Area 2: Operations and Algebraic Thinking**

For extended vignettes on how children represent and solve addition and subtraction problems, refer to the following:

*California Preschool Curriculum Framework, Volume 1* (Additional resources from the *Preschool Curriculum Frameworks* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "Bringing It All Together: Bagel Shop" page 256

*Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* (Transitional Kindergarten vignettes and snapshots from the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "How Do Students Learn to Add and Subtract Using Numbers Flexibly in Transitional Kindergarten Through Grade Two?" Chapter 3
- "Number Talk with Addition, Grade Two" Appendix C
- "Alex Builds Numbers with a Partner" (a two-day lesson) Appendix C
- "Students Examine and Connect Methods of Multiplication" Appendix C

**Key Area 3: Number and Operations in Base Ten**
For extended vignettes on how children use place value to add and subtract, refer to the following:

*Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve*

- "Number Talk with Addition, Grade Two" Appendix C
- "Alex Builds Numbers with a Partner" (a two-day lesson) Appendix C
- "Students Examine and Connect Methods of Multiplication" Appendix C

**Key Area 4: Measurement**
For extended vignettes on how children measure and estimate length, refer to the following:

*California Preschool Curriculum Framework, Volume 1* (Additional resources from the *Preschool Curriculum Frameworks* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "Mathematical Reasoning in Action: Which Is Taller?" page 275
- "Bringing It All Together: Tracking the Growth of Sunflowers" page 279

*Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve*

- "Content Connections—Reasoning with Data" Chapter 6
- "Habitat and Human Activity" Appendix C
- "Santikone Builds Rectangles to Find Area" Appendix C

**Key Area 5: Data**
For extended vignettes on how children classify, represent, and interpret data, refer to the following:

*California Preschool Curriculum Framework, Volume 1* (Additional resources from the *Preschool Curriculum Frameworks* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "Mathematical Reasoning in Action: Collecting Leaves on a Nature Walk" page 260
- "Bringing It All Together: Sorting, Counting, Graphing, and Comparing Apples" page 269

*Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve*

- "Logan's Early Elementary School Exploration with Data" Chapter 5
- "Content Connections—Reasoning with Data" Chapter 6
- "Habitat and Human Activity" Appendix C

**Key Area 6: Geometry and Spatial Thinking**

For extended vignettes on how children create, compose, and partition shapes, refer to the following:

*California Preschool Curriculum Framework, Volume 1* (Additional resources from the *Preschool Curriculum Frameworks* should be used alongside the current *Preschool/Transitional Kindergarten Learning Foundations*)

- "Mathematical Reasoning in Action: Discovering Shapes with Blocks" page 282
- "Bringing It All Together: Building a Castle" page 288

*Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve*

- "Santikone Builds Rectangles to Find Area" Appendix C

# Endnotes

1 Moira R. Dillon, Véronique Izard, and Elizabeth S. Spelke, "Infants' Sensitivity to Shape Changes in 2D Visual Forms," *Infancy* 25, no. 5 (September 2020), 618–639; Fei Xu and Elizabeth S. Spelke, "Large Number Discrimination in 6-Month-Old Infants," *Cognition* 74, no. 1 (January 2000), B1–B11.

2 National Research Council, *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity* (Washington, DC: The National Academies Press, 2009), 243–245, 332–337.

3 Mary Helen Immordino-Yang, Linda Darling-Hammond, and Christina Krone, *The Brain Basis for Integrated Social, Emotional, and Academic Development: How Emotions and Social Relationships Drive Learning* (Aspen Institute, 2018).

4 Alexis A. Lopez, Danielle Guzman-Orth, and Sultan Turkan, "Exploring the Use of Translanguaging to Measure the Mathematics Knowledge of Emergent Bilingual Students," *Translation and Translanguaging in Multilingual Contexts* 5, no. 2 (April 2019), 143–164; Aria Razfar, "¡Vamos a Jugar Counters! Learning Mathematics Through Funds of Knowledge, Play, and the Third Space," *Bilingual Research Journal* 35, no. 1 (May 2012), 53–75.

5 Jenna N. Futterer, Rebecca J. Bulotsky-Shearer, and Rinatte L. Gruen, "Emotional Support Moderates Associations Between Preschool Approaches to Learning and Academic Skills," *Journal of Applied Developmental Psychology* 80 (Spring 2022).

6 Irena Nayfeld, Janna Fuccillo, and Daryl B. Greenfield, "Executive Functions in Early Learning: Extending the Relationship Between Executive Functions and School Readiness to Science," *Learning and Individual Differences* 26 (August 2013), 81–88.

7 Emily Slusser, Andrew Ribner, and Anna Shusterman, "Language Counts: Early Language Mediates the Relationship Between Parent Education and Children's Math Ability," *Developmental Science* 22, no. 3 (November 2019).

8 National Research Council, *Mathematics Learning in Early Childhood*, 243–245, 332–337.

9 Luis C. Moll, Cathy Amanti, Deborah Neff, and Norma Gonzalez, *Funds of Knowledge* (Routledge, 2006), 71–87; DeAnn Huinker, Cathery Yeh, and Anne Marie Marshall, *Catalyzing Change in Early Childhood and Elementary Mathematics: Initiating Critical Conversations* (Reston, VA: National Council of Teachers of Mathematics, 2020), 45–68.

10 National Research Council, *Mathematics Learning in Early Childhood*, 332–337; National Academies of Sciences, *Engineering, and Medicine, Science and Engineering in Preschool Through Elementary Grades: The Brilliance of Children and the Strengths of Educators* (Washington, DC: The National Academies Press, 2022), 73–97.

11 Ozlem Cankaya, Natalia Rohatyn-Martin, Jamie Leach, Keirsten Taylor, and Okan Bulut, "Preschool Children's Loose Parts Play and the Relationship to Cognitive Development: A Review of the Literature," *Journal of Intelligence* 11, no. 8 (July 2023), 151; Robert S. Siegler and Geetha B. Ramani, "Playing Linear Numerical Board Games Promotes Low-Income Children's Numerical Development," *Developmental Science* 11, no. 5 (August 2008), 655–661; Constance Kamii, Yoko Miyakawa, and Yasuhiko Kato, "The Development of Logico-Mathematical Knowledge in a Block-Building Activity at Ages 1–4," *Journal of Research in Childhood Education* 19, no. 1 (November 2004), 44–57.

12 Raquel S. Klibanoff, Susan C. Levine, Janellen Huttenlocher, Marina Vasilyeva, and Larry V. Hedges, "Preschool Children's Mathematical Knowledge: The Effect of Teacher 'Math Talk'," *Developmental Psychology* 42, no. 1 (January 2006), 59–69; Geetha B. Ramani, Meredith L. Rowe, Sarah H. Eason, and Kathryn A. Leech, "Math Talk During Informal Learning Activities in Head Start Families," *Cognitive Development* 35 (Summer 2015), 15–33.

13 Victor Freiman, "Complex and Open-Ended Tasks to Enrich Mathematical Experiences of Kindergarten Student," in *Mathematical Creativity and Mathematical Giftedness: Enhancing Creative Capacities in Mathematically Promising Students*, Florence Mihaela Singer, ed. (Cham, Switzerland: Springer, 2018), 373–404; CAST, Universal Design for Learning Guidelines, version 3.0. (2024), http://udlguidelines.cast.org ; Rachel Lambert and Trisha Sugita, "Increasing Engagement of Students with Learning Disabilities in Mathematical Problem-Solving and Discussion," *Support for Learning* 31 (February 2016), 347–366; National Research Council, Mathematics Learning in Early Childhood, 243–245, 332–337.

14 Freiman, "Complex and Open-Ended Tasks to Enrich Mathematical Experiences of Kindergarten Student," 373–404; CAST, *Universal Design for Learning Guidelines*; Lambert and Sugita, "Increasing Engagement of Students with Learning Disabilities in Mathematical Problem-Solving and Discussion."

15 Moll et al., *Funds of Knowledge*, 71–87; Huinker, Yeh, and Marshall, *Catalyzing Change in Early Childhood and Elementary Mathematics*, 45–68.

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